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Planar Ramsey numbers for cycles

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Abstract

For two given graphs G and H the planar Ramsey number $\mathrm{PR}(G, H)$ is the smallest integer n such that every planar graph F on n vertices either contains a copy of G or its complement contains a copy H. By studying the existence of subhamiltonian cycles in complements of sparse graphs, we determine all planar Ramsey numbers for pairs of cycles.

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1. Introduction

Given two graphs G and H, *the Ramsey number* $R(G, H)$ is the smallest integer n such that every graph F on n vertices contains a copy of G or its complement contains a copy of H. The determination of Ramsey numbers is, in general, an extremely difficult problem. Often in such cases graph theorists turn to specific classes of graphs in hope for some positive results, and the class of planar graphs is one of the most attractive. This was apparently the reason why Walker [12] in 1969, and, independently (sic!), Steinberg and Tovey [11] in 1993 introduced the notion of planar Ramsey number.

The planar Ramsey number $\mathrm{PR}(G, H)$ is the smallest integer n for which every *planar* graph F on n vertices contains a copy of G or its complement contains a copy of H. Note that $\mathrm{PR}(G, H) \leqslant R(G, H)$, but unlike $R(G, H)$, as a simple consequence of Turán's Theorem, the numbers $\mathrm{PR}(G, H)$ grow only linearly with $|V(H)|$. Moreover, they do not need to be symmetric with respect to G and H. A related feature is that quite often all planar graphs F with $\mathrm{PR}(G, H)$ vertices, out of the two alternatives satisfy only the latter, i.e., $F \not\supset G$ but $F^c \supset H$. This is obviously true when G is not planar, but even for planar G it may be the case (see Theorem 6).

Both papers [12,11] focus on computing the planar Ramsey numbers for complete graphs and link them with the Four Color Conjecture, and, resp. Four Color Theorem. To pinpoint the values of these numbers, the authors of [11] use Grünbaum's Theorem, which generalizes the better known Grötzsch's Theorem. They show that $\mathrm{PR}(K_3, K_l) = 3l - 3$ and $\mathrm{PR}(K_k, K_l) = 4l - 3$ for all $k \geqslant 4$ and $l \geqslant 3$.

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In this paper we determine all planar Ramsey numbers for pairs of cycles, that is, all numbers $\mathrm{PR}(C_m, C_n)$, where $m \geqslant 3$ and $n \geqslant 3$. It turns out that for n large enough these numbers do not depend on m, while, on the other hand, for small n they do not grow with m, which emphasizes the asymmetry mentioned above. In our proofs we rely on well-known sufficient conditions for the existence of hamiltonian cycles and for pancyclicity. On the other hand, for some small cycles we make use of programs generating planar graphs [13].

Planar Ramsey numbers were also considered in [1,2]. For more on Ramsey numbers see, e.g., [8]. Throughout the paper we use the standard graph theory notation (see, e.g., [5]). In particular, $\delta(G)$, $\alpha(G)$ and $\kappa(G)$ stand, respectively, for the minimum degree, the independence number and the vertex-connectivity of a graph G. By G^c we denote the complement of G.

2. Cycles in complements of sparse graphs

The study of planar Ramsey numbers for cycles reduces, in most instances, to finding long cycles in complements of sparse, but not necessarily planar, graphs. In this section we present three similar results in this direction, all leading to the determination of the planar Ramsey numbers in question.

In our proofs we will frequently use the well-known sufficient conditions for a graph to be hamiltonian and to be pancyclic. All of them can be found, for instance, in the monograph [3]. Here is the list of what we will need.

- (Dirac) If $\delta(G) \geqslant n/2$ then G is hamiltonian.
- (Chvátal–Erdős) If $\alpha(G) \leqslant \kappa(G)$ then G is hamiltonian.
- (Bondy) If G is hamiltonian and $e(G) \geqslant n^2/4$ then G is pancyclic unless $G = K_{n/2,n/2}$.

Our first result asserts that there is a subhamiltonian cycle in each graph which, in some sense, is sparse both, locally and globally. As the example of $K_{3,3} - e$ shows, Theorem 1 does not need to be true for $n \leqslant 6$. Also, it fails to be true if we instead request a copy of C_{n-1} in the complement. Here $K_{2,n-2}$ is a counterexample.

Theorem 1. *Let G be an arbitrary graph on $n \geqslant 7$ vertices, at most* $\max\{\binom{n}{2} - n^2/4, 2n - 4\}$ *edges and containing neither K_3 nor $K_{3,3}$. Then G^c contains C_{n-2}.*

Proof. We have $\alpha(G^c) \leqslant 2$. If $\kappa(G^c) \geqslant 2$, then G^c is hamiltonian by the Chvátal–Erdős condition. For $n \geqslant 8$, we have $e(G^c) \geqslant n^2/4$ and $G^c \neq K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Thus G^c is pancyclic and in particular contains C_{n-2}. For $n = 7$, if $e(G^c) < 7^2/4$ then $e(G^c) = 11$ or 12, so there must be at least four diagonals in the C_7, and thus there is a C_5 in G^c.

If $\kappa(G^c) \leqslant 1$, then there is a vertex x which separates a nonempty set of vertices U from the rest. Note that $|U| \leqslant 2$, since G does not contain $K_{3,3}$. Set $W = V - U - \{x\}$. Because G is K_3-free, $G^c[W]$ must be complete. If $|U| = 1$ then we have C_{n-2} in G^c, so consider $|U| = 2$. Since $|W| \geqslant 4$, there are at least two edges from x to W (otherwise there would be $K_{3,3}$ in G). So, $W \cup \{x\}$ spans a C_{n-2}. □

The next result is not new. Nevertheless, it fits well with the other theorems here, and we give the proof because of its simplicity. Together with a lower bound given by the star $K_{1,n-1}$, it yields that $\mathrm{PR}(C_4, C_n) = n + 1$.

Theorem 2 (*[6,9,10]*). *For $n \geqslant 6$, the Ramsey number $R(C_4, C_n) = n + 1$. In other words, the complement of every C_4-free graph on $n \geqslant 7$ vertices contains C_{n-1}.*

Proof. Let G be a graph fulfilling the assumptions of the theorem. We have $\alpha(G^c) \leqslant 3$. If $\kappa(G^c) \geqslant 3$ then, by the Chvátal–Erdős condition, G^c is hamiltonian. It is easy to see that for $n \geqslant 8$, the Turán number $T(n, C_4)$ is smaller than $\binom{n}{2} - n^2/4$ (use the well-known bound $T(n, C_4) \leqslant \frac{1}{4}n(1 + 2\sqrt{n})$—see, e.g., [8, p. 144]), and thus G^c is pancyclic, containing, in particular, C_{n-1}. For $n = 7$, $T(n, C_4) = 9$ (see [4]) and $\binom{n}{2} - n^2/4 = 8.75$, but it can be checked by hand that there is no C_4-free graph on seven vertices whose complement is hamiltonian.

If $\kappa(G^c) \leqslant 2$ then there are two vertices x, y which separate one vertex, a, from the rest. Set $W = V - \{a, x, y\}$. Because G is C_4-free, every vertex of W has in G^c at least $n - 5$ neighbors in W, while x and y have each at least $n - 4$ neighbors in W. Hence, by Dirac's condition, $G^c[W]$ has a Hamilton cycle which can be extended, by adding x and y, to a cycle C_{n-1}. □