Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 4460–4472
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Problems and results in extremal combinatorics—II☆

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Received 13 January 2004; accepted 8 August 2007
Available online 29 October 2007

Dedicated to Miki Simonovits, for his 60th birthday

Abstract

Extremal Combinatorics is one of the central areas in Discrete Mathematics. It deals with problems that are often motivated by questions arising in other areas, including Theoretical Computer Science, Geometry and Game Theory. This paper contains a collection of problems and results in the area, including solutions or partial solutions to open problems suggested by various researchers. The topics considered here include questions in Extremal Graph Theory, Polyhedral Combinatorics and Probabilistic Combinatorics. This is not meant to be a comprehensive survey of the area, it is merely a collection of various extremal problems, which are hopefully interesting. The choice of the problems is inevitably biased, and as the title of the paper suggests, it is a sequel to a previous paper [N. Alon, Problems and results in extremal combinatorics—I, Discrete Math. 273 (2003), 31–53.] of the same flavor, and hopefully a predecessor of another related future paper. Each section of this paper is essentially self contained, and can be read separately.

Keywords: Extremal Graph Theory; Coupon collector; Covering codes

1. Introduction

Extremal Combinatorics deals with the problem of determining or estimating the maximum or minimum possible value of an invariant of a combinatorial object that satisfies certain requirements. Problems of this type are often related to other areas including Computer Science, Information Theory, Number Theory and Game Theory. This branch of Combinatorics has been very active during the last few decades, see, e.g., [5,10], and their many references.

We discuss several problems and results in the area, including solutions or partial solutions to open problems suggested by various researchers. The questions considered include problems in Extremal Graph Theory, Polyhedral Combinatorics and Probabilistic Combinatorics. This is not a comprehensive survey of the area, but rather a collection of several extremal problems. The techniques used include combinatorial, probabilistic and algebraic tools. Each section of this paper is essentially self contained, and can be read separately.

☆ Research supported in part by a USA-Israeli BSF grant, by the Israel Science Foundation and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University.
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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.08.090

2. Balanced subgraphs in dense graphs

As mentioned below the title, this paper is dedicated to Miki Simonovits, one of the founders of modern Extremal Graph Theory. It is therefore natural to start with a problem he raised, together with Erdős, more than 30 years ago in [7]. A graph is called *D-balanced* if the ratio between the maximum degree of a vertex in it and the minimum degree of a vertex in it is at most D. In [7] Erdős and Simonovits prove that for every fixed $\alpha > 0$, there exists a finite $D = D(\alpha)$ such that if $n > n_0(\alpha)$, then any graph with n vertices and at least $n^{1+\alpha}$ edges contains a D-balanced subgraph with m vertices and at least $\frac{2}{5}m^{1+\alpha}$ edges, where $m \geqslant n^{\alpha(1-\alpha)/(1+\alpha)}$. They apply this fact to prove, among other things, that the maximum number of edges in a graph on n vertices that contains no copy of the graph of the 3-cube is at most $O(n^{8/5})$—a result that is not known to be tight but is still the best known upper estimate for this question.

They conclude their paper by raising the following question.

Problem (*Erdős-Simonovits [7]*). Is it true that there are absolute constants $\varepsilon > 0$ and D, such that the following holds: For every m there is some $n_0 = n_0(m)$ such that any graph with $n > n_0$ vertices and at least $n \log_2 n$ edges contains a D-balanced subgraph with m vertices and at least $\varepsilon m \log_2 m$ edges?

In this section we show that this is not true. The proof is probabilistic, and is based on a modification of the technique of Pyber et al., who proved in [14] that there are graphs with n vertices and $\Omega(n \log\log n)$ edges that contain no 3-regular subgraphs. For simplicity of presentation, we omit in this section all floor and ceiling signs, whenever these are not crucial. We make no attempt to optimize the various absolute constants that appear in the proof. All logarithms in this section are in base 2.

Proposition 2.1. *For every $D > 1$ and every $n > 10^5$, there is a graph G with at most $2n$ vertices and at least $2n\log(2n)$ edges such that the following holds. For any m and d, if there is a subgraph H of G with m vertices, average degree at least d, and maximum degree at most Dd, then $d < 36(4\sqrt{\log m} + \log(64D) + 18)$.*

Proof. Let G be a random bipartite graph constructed as follows. Let A be a set of n vertices. For each i satisfying $4 \leqslant i \leqslant \frac{1}{2}\log n$, and each j satisfying $1 \leqslant j \leqslant 8$, let $B_{i,j}$ be a set of $n/2^i$ vertices, where all the sets A and $B_{i,j}$ are pairwise disjoint. Let $B = \bigcup_{i,j} B_{i,j}$ be the union of all sets $B_{i,j}$. The two classes of vertices of G are the sets A and B. The edges of G are chosen randomly; for each i, $4 \leqslant i \leqslant \frac{1}{2}\log n$, and for each j, $1 \leqslant j \leqslant 8$, each vertex of A is connected to one random, uniformly chosen vertex of $B_{i,j}$, where all choices are independent. Note that $|A| = n$, $|B| = 8\sum_{i=4}^{\log n/2} n/2^i < n$ and the degree of each vertex of A is $8(\frac{1}{2}\log n - 3)$ (which is larger than $2\log(2n)$ for $n > 10^5$).

Claim. *The following holds almost surely (that is, with probability that tends to 1 as n tends to infinity). Let $m \leqslant n$ and r be positive integers satisfying $r \geqslant 4\sqrt{\log m} + 16$ and suppose $A' \subset A$ and $B' \subset B$ satisfy $|A'| \leqslant m$, $|B'| \leqslant m$. Suppose, further, that $B' \subset \bigcup_{i\in I}\bigcup_{1\leqslant j\leqslant 8} B_{i,j}$, where*

$$I = \left\{ i : 4 \leqslant i \leqslant \frac{1}{2}\log n,\ \frac{n}{2^i} \geqslant m2^r \right\}.$$

Then the number of edges of the induced subgraph of G on $A' \cup B'$ is smaller than mr.

Proof of Claim. It suffices to show that almost surely there are no subsets $A' \subset A$, $B' \subset B$ of cardinality precisely m each, that span at least mr edges (since we can always add vertices to these sets, if needed). There are less than $\binom{n}{m}^2$ ways to choose two such sets A', B'. Having chosen these sets, there are $\binom{m^2}{mr}$ ways to choose a set of mr edges connecting a vertex of A' with one of B'. For each such fixed choice of edges, the probability that all of them are edges of the random graph G is zero in case two of these edges connect the same vertex of A' to two distinct members of the same set $B_{i,j}$. In any other case, this probability is at most $(1/m2^r)^{mr}$, and also at most $(1/\sqrt{n})^{mr}$. Indeed, each of these edges is chosen as an edge of G with probability $1/|B_{i,j}|$ for some $i \in I$, and all these choices are mutually independent. Since $|B_{i,j}| \geqslant \max(\sqrt{n}, m2^r)$ for each $i \in I$, the above two estimates indeed hold. It therefore follows