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ScienceDirect
Discrete Mathematics 307 (2007) 769–773
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A note on the structure of spaces of domino tilings

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Received 2 August 2005; received in revised form 5 May 2006; accepted 3 July 2006
Available online 29 September 2006

Abstract

We study spaces of tilings, formed by tilings which are on a geodesic between two fixed tilings of the same domain (the distance is defined using local flips). We prove that each space of tilings is homeomorphic to an interval of tilings of a domain when flips are classically directed by height functions.

Keywords: Domino tilings; Flips; Lattice; Order theory

1. Introduction

In this paper, we study tilings of finite regions of the square lattice with *dominoes* (i.e. 2×1 rectangles). These tilings are of particular importance in theoretical physics, where a domino is seen as a dimer, a diatomic molecule (as the molecule of dihydrogen) and each tiling is a possible state of a solid, or a fluid.

Flips (i.e. local transformations involving only two tiles covering a 2×2 square) induce a dynamics on tilings which has a central role in tiling theory. Flips can be directed using *height functions*, introduced by Thurston [16]. A central result is that, given a hole-free domain D, the directed graph formed on tilings of D, where edges correspond to upwards flips, is the covering relation of a distributive lattice [2,14] (we refer to [4] for lattice theory). Some extensions for figure with holes can be found in [12,7].

These structural results have some algorithmic applications. They are used by Luby et al. [11], and Wilson [17] to obtain a rapidly mixing monotonic Markov process to sample tilings uniformly at random. Desreux and Rémila also use the structure to obtain listing algorithms [6,8].

In this note, we explore another way of directing flips: given a fixed tiling T_0, flips are directed in such a way that the origin of the flip is closer to T_0 than the tail of the flip. Precisely, given another tiling T_1 such that T_0 and T_1 are linked by a sequence of (upwards and downwards) flips, we study the directed graph formed by tilings T such that $d(T_0, T) + d(T, T_1) = d(T_0, T_1)$ and edges are directed as indicated above. Our result is that the spaces of tilings obtained by this way are all isomorphic to intervals of spaces of tilings obtained previously, using upwards flips. Therefore, they inherit the same structural and algorithmic properties.

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doi:10.1016/j.disc.2006.07.022

We focus on spaces of dominos tilings, but a similar result can be obtained on lozenge (or calisson) tilings, in a parallel way. This is a classical paradigm: domino tilings and lozenge tilings have the same properties. Lozenge tilings are a particular case of the class of rhombus tilings. Those tilings are defined using two parameters: the dimension d of the Euclidean space, and the number n of vectors allowed for supporting the edges of rhombic tiles (we have $n = 3$ and $d = 2$ for what is usually called lozenge tilings). The complexity of the space of tilings seems to be directly linked to the codimension $c = n - d$. For $c = 1$, the space of tilings is a distributive lattice. For $c = 2$, the space of tilings of a fixed zonotopal domain is a graded poset, but the lattice structure is lost in the general case. Latapy [10] introduced a recursive approach of these tilings, by seeing a tiling using $n + 1$ vectors as a solution of a partition problem on a tiling using n vectors. For $c \geqslant 3$, rhombic tilings remain very mysterious, even if some new methods have proved the structure of graded poset in some particular cases [1,5]. The question of the connectivity of such spaces is still open in the general case, and even for the particular case when $n = 6$ and $d = 3$.

On the other hand, structures of distributive lattices also appear for bar tilings [9,13], which are a natural generalization of domino tilings, where the length of the largest side is a parameter m (for dominoes, we have $m = 2$). The connectivity of the space of tilings with two rectangles (using two kings of flips) has also been recently proved [15], but no more is known about the structure of such spaces of tilings.

2. Tilings and flips

Let Λ be the planar grid of the Euclidean plane $\mathbb{R}^2$. A *vertex* of Λ is a point with both integer coordinates. Let $v = (x, y)$ be a vertex of Λ. A *cell* of Λ is a (closed) unit square whose corners are vertices. Two vertices of Λ are *neighbors* if they are both ends of the same side of a cell of Λ. Hence, each vertex v has four neighbors. An edge is a pair (v, v) of neighbor vertices.

We assume that cells of Λ are colored as a checkerboard. By this way, we have black cells and white cells, and two cells sharing a side have different colors. For each edge (v, v') of Λ, we define the *spin* of (v, v') (denoted by $sp(v, v')$) by

- $sp(v, v') = 1$ if an ant moving from v to v', (following the line segment $[v, v']$) has a white cell on its left side (and a black cell on its right side),
- $sp(v, v') = -1$ otherwise.

A *figure* F of Λ is a finite union of cells of Λ. The set of vertices of cells of F is denoted by V_F. The set of edges of F (denoted by E_F) is the set of ordered pairs (v, v') of $(V_F)^2$ such that the line segment $[v, v']$ is a side of a cell of F.

A *domino* is a figure formed from two cells with a common side, which is called the *central axis* of the domino. A *tiling* T of a figure F is a set of dominoes included in F, with pairwise disjoint interiors (i.e. there is no overlap), such that the union of tiles of T equals F (i.e. there is no gap).

A *local flip* (see Fig. 1) is the replacement in T of the pair of dominoes which cover a 2×2 square by the other pair which can cover S. Let v denote the central vertex of the square, a new tiling T_v is obtained by this replacement. We say that T_v is obtained from T by a flip around v.

Two tilings such that one can be obtained from the other one by a single flip are *neighbors*. A *path of tilings* is a sequence $(T_0, T_1, \ldots, T_p)$ of tilings such that for each integer i, with $0 \leqslant i < p$, T_{i+1} and T_i are neighbors. The integer p is the *length* of the path.

Two tilings, T and T' are *connected by flips* if there exists a path of tilings linking T and T'. In this case, the *flip distance* $d(T, T')$ is the minimal number of successive flips necessary to transform T into T'. A path linking T and T' of length $d(T, T')$ is called a *geodesic*.

Using the spin, flips can be directed as follows: we say that the flip is *going upwards* if, in the tiling T, v is the center of a long side $[v', v'']$ shared by two dominos, and $sp(v, v') = sp(v, v'') = 1$; otherwise, we say that the flip is *going downwards*.

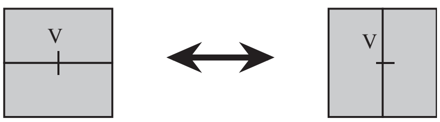

Fig. 1. A local flip.