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On the sphericity testing of single source digraphs

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Abstract

A digraph $D = (V, A)$ is called spherical, if it has an upward embedding on the round sphere which is an embedding of D on the round sphere so that all edges are monotonic arcs and all point to a fixed direction, say to the north pole. It is easy to see that [S.M. Hashemi, Digraph embedding, Discrete Math. 233 (2001) 321–328] for upward embedding, plane and sphere are not equivalent, which is in contrast with the fact that they are equivalent for undirected graphs. On the other hand, it has been proved that sphericity testing for digraphs is an *NP-complete* problem [S.M. Hashemi, A. Kisielewicz, I. Rival, The complexity of upward drawings on spheres, Order 14 (1998) 327–363]. In this work we study sphericity testing of the single source digraphs. In particular, we shall present a polynomial time algorithm for sphericity testing of three connected single source digraphs.

Keywords: Embedding; Upward embedding; Sphericity; Planarity; Upward planarity; Single source digraph

1. Introduction

A digraph $D = (V, A)$ is called spherical, if it has an *upward embedding* on the round sphere $S = \{(x, y, z): (x^2 + y^2 + z^2) = 1\}$ which is an embedding of D on the S so that all edges are monotonic arcs and all point to a fixed direction, say to the north pole. A necessary condition for an acyclic directed graph to be spherical is that its underlying graph to be planar—it has an embedding on the plane. Although there are some literatures on upward planarity of single source digraphs [8,7,3], in which there are polynomial-time algorithms for that problem, it has been proved that [5] for upward embedding, plane and sphere are not equivalent which is in contrast with the fact that they are equivalent for undirected graphs. So clearly the same methods of proof for upward planarity testing cannot be applied to the sphericity testing.

In Fig. 1(a) we have presented an upward drawing of a single source digraph on the plane. Fig. 1(b) illustrates an embedding of its underlying graph on the plane. This digraph has no upward embedding on the plane. However, this digraph is spherical. An upward embedding of this digraph on the sphere has been illustrated in Fig. 2.

In this paper we shall show that there is a polynomial time algorithm for sphericity testing of three connected single source digraphs. The remainder of this paper is organized as follows. A combinatorial characterization of spherical digraphs in terms of critical points is presented in Section 2. In Section 3 we present a polynomial time algorithm to

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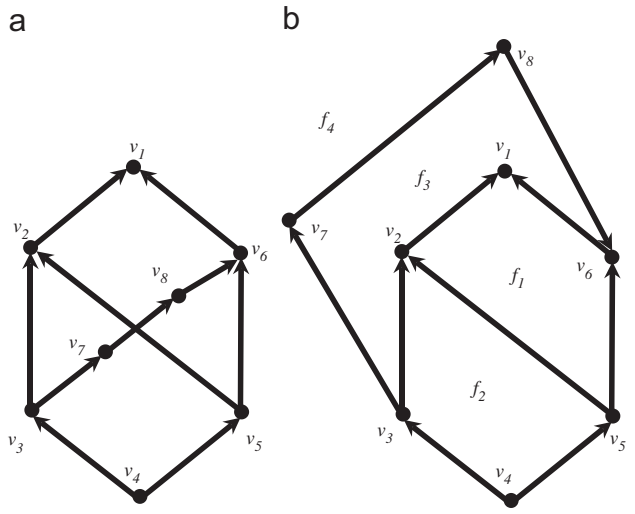

Fig. 1. (a) A single source digraph; (b) an embedding of it on the plane.

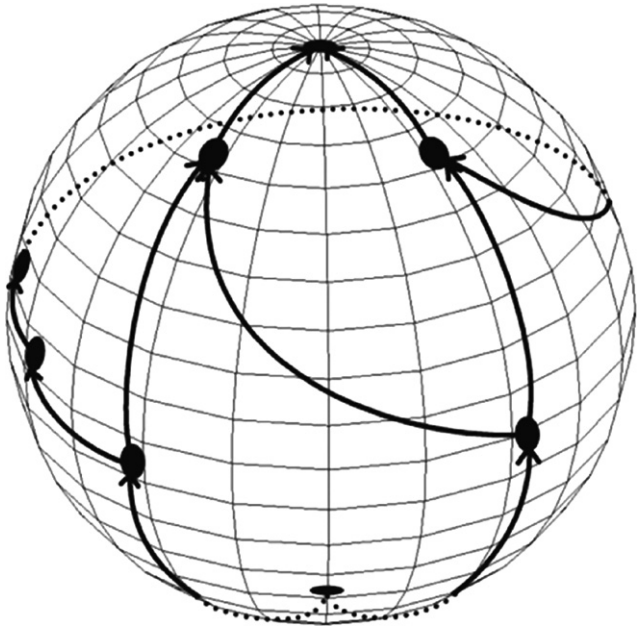

Fig. 2. An upward embedding of a single source digraph on the round sphere.

determine whether a single source digraph with a given embedding on the plane has a similar upward embedding on the sphere. We present the proof of the results in Section 4.

2. Spherical digraph

In this section we summarize a characterization of spherical digraphs based on [5]. The pattern of incoming edges and outgoing edges incident upon vertices are important for describing spherical digraphs. We first introduce the notion of critical vertices in a given embedding of a digraph on the plane. Then based on these notions a characterization of spherical digraphs will be presented.