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On minimally circular-imperfect graphs ☆

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Abstract

A circular-perfect graph is a graph of which each induced subgraph has the same circular chromatic number as its circular clique number. In this paper, (1) we prove a lower bound on the order of minimally circular-imperfect graphs, and characterize those that attain the bound; (2) we prove that if G is a claw-free minimally circular-imperfect graph such that $\omega_c(G-x)>\omega(G-x)$ for some $x\in V(G)$, then $G=K_{(2k+1)/2}+x$ for an integer k; and (3) we also characterize all minimally circular-imperfect line graphs.

MSC: 05c15; 05c78

Keywords: Circular coloring; Circular clique; Circular-perfect

1. Introduction

All graphs considered are finite and simple, i.e., finite graphs without multiedges and loops. Undefined concepts and terminologies are from [5].

Let $G=(V,E)$ be a graph, where V and E denote the vertex set and edge set of G, respectively. Two vertices u and v are adjacent, denoted by $uv\in E(G)$, if there is an edge in $E(G)$ joining them. A proper subgraph of G is a subgraph which is not G itself. A subgraph H of G is called an *induced subgraph* if $E(H)=\{uv|u\in V(H), u\in V(H),$ and $uv\in E(G)\}$. Let $S\subset V$ be a subset of vertices. We use $G[S]$ to denote the subgraph of G induced by S.

A graph G is called a *perfect graph* if every induced subgraph H of G has the same chromatic number $\chi(H)$ as its clique number $\omega(H)$. A *minimally imperfect graph* is an imperfect graph of which each proper induced subgraph is perfect. An *odd hole* is an odd circuit of length at least five. The famous *Perfect Graph Conjecture* [3] was proved by Chudnovsky et al. in [6].

Theorem 1 (*Chudnovsky et al. [6], Strong Perfect Graph Theorem*). *The only minimally imperfect graphs are the odd holes and their complements.*

Let k and d be positive integers with $k\geqslant 2d$. A (k,d)-*circular coloring* of a graph G is a mapping $\psi: V(G)\mapsto\{0,1,2,\ldots,k-1\}$ such that $d\leqslant|\psi(u)-\psi(v)|\leqslant k-d$ whenever $uv\in E(G)$. A graph G is called k/d-circular

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colorable if it admits a (k, d)-circular coloring. The *circular chromatic number* of G, denoted by $\chi_c(G)$, is defined as $\chi_c(G) = \inf\{k/d \mid G \text{ is } k/d\text{-circular colorable}\}$.

The concept of circular coloring was first introduced in 1988 by Vince [9] with the name *star coloring*, and it got the current name from Zhu [15]. It was proved elsewhere [4,9] that $\chi_c(G)$ is always attained at rational number and

$$\chi(G) - 1 < \chi_c(G) \leqslant \chi(G) \quad \text{for any graph } G. \tag{1}$$

A (k, d)-*partition* of G is a partition $(V_0, V_1, V_2, \ldots, V_{k-1})$ of $V(G)$ such that for each i, $0 \leqslant i \leqslant k-1$, $V_i \cup V_{i+1} \cup \cdots \cup V_{i+d-1}$ is an independent set of G, where the addition of indices is taken mod k ($V_i = \emptyset$ for some i is permitted). It is easy to see that a (k, d)-*partition* of G is simply the color classes of a (k, d)-*coloring* of G. Below is a theorem from [7].

Theorem 2 (*Fan [7]*). *A graph G has a (k, d)-circular coloring iff it has a (k, d)-partition. Furthermore, $\chi_c(G) = k/d$ iff G is k/d-circular colorable and for every (k, d)-partition $V_0, V_1, \ldots, V_{k-1}$ of G, $V_i \neq \emptyset$ for every i.*

Section 2 is devoted to the concept of circular-perfect graphs and some examples of minimally circular-imperfect graphs. In Section 3, we present a lower bound on the order of minimally circular-imperfect graphs, and characterize those that attain the bound. In Section 4, we characterize the claw-free minimally circular-imperfect graphs with the property that $\omega_c(G - x) > \omega(G - x)$ for some $x \in V(G)$. In the last section, we characterize all minimally circular-imperfect line graphs.

2. Circular-perfect graphs

Given two positive integers k and d with $k \geqslant 2d$, let $K_{k/d}$ be a graph with $V(K_{k/d}) = \{v_0, v_1, v_2, \ldots, v_{k-1}\}$ and $E(K_{k/d}) = \{v_i v_j \mid d \leqslant |j - i| \leqslant k - d\}$. While $d = 1$, $K_{k/d}$ is simply the complete graph K_k of order k.

It was proved that $\chi_c(K_{k/d}) = k/d$ [4,9]. So, if a graph G contains a subgraph H isomorphic to $K_{k/d}$ (we simply denote it by $H = K_{k/d}$) for some k and d, then $\chi_c(G) \geqslant k/d$. Unless otherwise specified, $\{v_0, v_1, v_2, \ldots, v_{k-1}\}$ and $\{v_i v_j \mid d \leqslant |j - i| \leqslant k - d\}$ always refer to the vertex set and edge set of $K_{k/d}$, respectively.

The *circular clique number* of G (first introduced by Zhu in [16]), denoted by $\omega_c(G)$, is defined as the maximum fractional k/d such that $K_{k/d}$ admits a homomorphism to G. Let $\gcd(k, d)$ be the greatest common divisor of integers k and d. Zhu proved in [16] that

Theorem 3 (*Zhu [16]*). *For any graph G,*

$$\omega(G) \leqslant \omega_c(G) < \omega(G) + 1 \tag{2}$$

and $\omega_c(G) = k/d$ for some k and d with $\gcd(k, d) = 1$ indicates that G contains an induced subgraph isomorphic to $K_{k/d}$.

A graph G is called *circular-perfect* if $\omega_c(H) = \chi_c(H)$ for each induced subgraph H of G [16]. Up to now, we do not know too much on the structure of circular-perfect graphs. Some sufficient conditions and necessary conditions for a graph to be circular-perfect were discussed in [11,16]. Bang-Jensen and Huang presented in [1] a family of circular-perfect graphs, they called them *convex-round graphs*, which is a super-family of $K_{k/d}$'s.

Theorem 4 (*Bang-Jensen and Huang [1], Zhu [16]*). *For any integers $k \geqslant 2d$, $K_{k/d}$ is circular-perfect.*

A *circular-imperfect* graph is a graph that is not circular-perfect, and a *minimally circular-imperfect graph* is a circular-imperfect graph of which each proper induced subgraph is circular-perfect. The Strong Perfect Graph Theorem and Theorem 4 tell us that every minimally imperfect graph is in fact circular-perfect.

To study the circular-perfect graphs, a natural approach is to characterize the minimally circular-imperfect graphs. It seems that the structure of minimally circular-imperfect graphs is much more complicated than that of