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On the tightness of the $\frac{5}{14}$ independence ratio

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Abstract

In 1979, Staton proved that every triangle-free graph G with maximum degree at most three has an independent set with size at least $\frac{5}{14}$ of the number of vertices of G. Fraughnaugh [size and independence in triangle-free graphs with maximum degree three, J. Graph Theory 14 (5) (1990) 525–535] and Heckman and Thomas [A new proof of the independence ratio of triangle-free cubic graphs, Discrete Math 233 (2001) 233–237] provided shorter proofs of the same result. An analysis of the cases of equality for the main results in the last paper is presented. Also, a proof that there are only two connected triangle-free graphs with maximum degree at most three and independence ratio $\frac{5}{14}$ is given; it is self-contained and does not require a computer search.

Keywords: Independent set; Independent ratio; Stable set; Triangle-free

1. Introduction

Graph terms which are not defined in this paper use the same definitions as in [3], or any other standard graph theory textbook. We will also use $n(G)$, $e(G)$, and $\alpha(G)$ to denote, in order, the number of vertices of G, the number of edges of G, and the independence number of G, omitting the (G) when there is no danger of confusion.

A graph G is said to be *triangle-free* if no subgraph of G is isomorphic to the complete graph K_3. An *independent set* is a set of vertices, no two of which are adjacent, and the *independence number of* G is the size of a largest independent set. The *independence ratio of* G is defined to be the independence number of G divided by the number of vertices in G.

All graphs mentioned in this paper will be assumed to be simple, loopless, triangle-free, and have maximum degree at most three, unless explicitly stated otherwise.

In 1979, Staton proved:

Theorem 1.1 (*Staton [9]*). *The independence ratio of a triangle-free graph with maximum degree at most three is at least* $\frac{5}{14}$.

This settled a conjecture by Albertson et al. [1]. A shorter proof was found later by Fraughnaugh [7] and an even shorter one by Heckman and Thomas [6]. The constant $\frac{5}{14}$ is the best possible because, as noted by Fajtlowicz [4], the generalized Petersen graph $P(7, 2)$ has 14 vertices, no triangles, and no independent set with size six.

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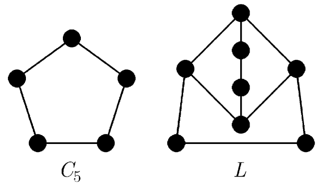

Fig. 1. Difficult blocks.

The question remained of whether there were any other connected triangle-free graphs with maximum degree at most three with an independence ratio of exactly $\frac{5}{14}$. One was found by Locke [8], and a computer search performed by Bajnok and Brinkmann [2] showed that these are the only two triangle-free graphs with maximum degree three having 14 vertices and independence number five. Both of these graphs are shown in Fig. 5.

The question of whether there are any larger connected graphs with an independence ratio of $\frac{5}{14}$ was settled by Fraughnaugh and Locke, who showed that the constant $\frac{5}{14}$ is not best possible in an asymptotic sense; they showed:

Theorem 1.2 (*Fraughnaugh and Locke [5]*). *The independence number of every connected triangle-free graph with maximum degree at most three is at least* $\frac{11}{30}n - \frac{2}{15}$.

This bound is better than $\frac{5}{14}n$ when G has more than 14 vertices; this result, combined with Bajnok and Brinkmann's result, show that there are no other connected graphs with an independent ratio of exactly $\frac{5}{14}$.

This settled the problem, but since [2] uses a computer search, the result is open to question. This paper will provide a self-contained proof that there are only two connected graphs with an independence ratio of exactly $\frac{5}{14}$, without having to do any exhaustive computer searches.

The main result from Bajnok and Brinkmann is a characterization of equality of the main result in [7], which is the following (**M** is a particular set of graphs defined in Fraughnaugh's paper):

Theorem 1.3 (*Fraughnaugh, [5]*). *If G is a triangle-free graph with maximum degree at most three, then* $e \geqslant \frac{13}{2}n - 14\alpha$. *Moreover, one of the following holds*:

(i) $e \geqslant \frac{13}{2}n - 14\alpha + 3$;
(ii) *G contains K_2 as a component and* $e \geqslant \frac{13}{2}n - 14\alpha + 2$;
(iii) *G has minimum degree equal to* 2 *and G contains a* 4-*cycle*;
(iv) *G is one of the graphs in* **M**;
(v) *G is* 3-*regular.*

Bajnok and Brinkmann [2] showed that equality in Theorem 1.3 occurs for exactly three connected graphs: the two graphs with 14 vertices mentioned above, and the graph L (shown in Fig. 1) consisting of eight vertices and 10 edges, which has an independence number of three.

Heckman and Thomas [6] were inspired by Fraughnaugh's proof, and looked at ways of making it more efficient. The main obstacles to a short proof were called *difficult blocks*, which turn out to be the pentagon (C_5) and the graph L shown in Fig. 1.

A graph G is called *difficult* if, after deleting all the cut-edges of G, every component is a difficult block. The number $\lambda(G)$ is defined to be the number of components of G which are difficult. Then the following holds:

Theorem 1.4 (*Heckman and Thomas, [6]*). *The independence number of every triangle-free graph with maximum degree at most three is at least* $\frac{4}{7}n - \frac{1}{7}e - \frac{1}{7}\lambda(G)$.

To see why Theorem 1.4 implies the $\frac{5}{14}$ independence ratio, consider a connected graph G. If G is difficult, then its independence ratio is at least $\frac{3}{8}$ (and equality is attained if every block of G is L or a single edge). Otherwise,

$$\alpha(G) \geqslant \frac{4}{7}n - \frac{1}{7}e = \frac{5}{14}n + \frac{1}{14}(3n - 2e) = \frac{5}{14}n + \sum_{v \in V(G)} (3 - \deg v) \geqslant \frac{5}{14}n.$$