Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 3241–3248
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Domination dot-critical graphs with no critical vertices☆

Zhao Chengye, Yang Yuansheng*, Sun Linlin

Department of Computer Science, Dalian University of Technology, Dalian, 116024, PR China

Received 1 June 2006; received in revised form 24 May 2007; accepted 24 June 2007
Available online 8 August 2007

Abstract

A graph G is dot-critical if contracting any edge decreases the domination number. It is totally dot-critical if identifying any two vertices decreases the domination number. Burton and Sumner [Discrete Math. 306 (2006) 11–18] posed the problem: Is it true that for $k \geqslant 4$, there exists a totally k-dot-critical graph with no critical vertices? In this paper, we show that this problem has a positive answer.

Keywords: Domination; Dot-critical; Critical vertex

1. Introduction

We only consider finite connected and undirected graphs without loops or multiple edges.

The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are denoted by $N(v) = \{u \in V(G): vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V(G)$ is a *dominating* set if and only if $N[S] = V(G)$. If S has the smallest possible cardinality of any dominating set of G, then S is called a *minimum dominating* set—abbreviated MDS. The cardinality of any MDS for G is called the *domination number* of G and is denoted by $\gamma(G)$. We denote the complement of the graph G by $\overline{G}$. We write $d(v, u)$ for the distance between the vertices v and u. A vertex v of G is *critical* if $\gamma(G - v) < \gamma(G)$. We denote the set of critical vertices of G by G'.

Burton and Sumner [1] introduced a new critical condition for the domination number. For a pair of vertices v, u of G, they denote by $G.vu$ the graph obtained by identifying v and u. So $G.vu$ may be viewed as the graph obtained from G by deleting the vertices v and u and appending a new vertex, denoted by (vu), that is adjacent to all the vertices of G-v-u that were originally adjacent to either of v or u. A graph G is *domination dot-critical* (hereafter, just *dot-critical*) if $\gamma(G.vu) = \gamma(G) - 1$ for any two adjacent vertices v and u. A graph G is *domination totally dot-critical* (hereafter, just *totally dot-critical*) if $\gamma(G.vu) = \gamma(G) - 1$ for any two vertices v and u.

When we say that a graph G is *k-dot-critical* or *totally-k-dot-critical*, we mean that it has the indicated property and that $\gamma(G) = k$.

☆ Research supported by NFSC (60573022).

* Corresponding author.
E-mail address: yangys@dlut.edu.cn (Y. Yuansheng).

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.06.029

The *corona* of two graphs H_1 and H_2 is the graph $H_1 \circ H_2$ formed from one copy of H_1 and $|V(H_1)|$ copies of H_2, where the ith vertex of H_1 is adjacent to every vertex in the ith copy of H_2, see [2].

A graph G is said to be *spiked* if $G = H \circ K_1$, the corona of a connected graph H with a single vertex. So G is spiked if it is non-trivial, connected, and every vertex of G is either an end vertex or is adjacent to exactly one end vertex.

Burton and Sumner [1] investigated some properties of the dot-critical (*resp*. totally dot-critical) graphs, characterized 2-dot-critical and totally 2-dot-critical graphs and showed the diameter of 3-dot-critical and totally 3-dot-critical graphs G with $G' = \emptyset$. They showed following theorems.

Theorem 1.1. *Let G be a graph with $n \geqslant 4$ vertices. Then G is 2-dot-critical if and only if $\overline{G}$ is not complete, but every component of $\overline{G}$ is spiked or a complete graph K_m, $m \geqslant 2$.*

Theorem 1.2. *G is a totally 2-dot-critical graph with $n \geqslant 2$ vertices if and only if every component of $\overline{G}$ is spiked.*

Theorem 1.3. *A connected 3-dot-critical graph G with $G' = \emptyset$ has a diameter of at most three.*

Theorem 1.4. *A connected totally 3-dot-critical graph G with $G' = \emptyset$ has a diameter of at most two.*

In the end, Burton and Sumner posed some open problems:

(1) What are the best bounds for the diameter of a k-dot-critical graph and a totally k-dot-critical graph G with $G' = \emptyset$ for $k \geqslant 4$?
(2) Is it true that for each $k \geqslant 4$, there exists a totally k-dot-critical graph with no critical vertices?

In this paper, we study the k-dot-critical graph and totally k-dot-critical graph G with $G' = \emptyset$ for $k \geqslant 4$. In Section 2, we prove that there exists a totally k-dot-critical graph with no critical vertices for each $k \geqslant 4$. In Section 3, we prove that a 4-dot-critical (*resp*. totally 4-dot-critical) graph has a diameter of at most five and this bound is sharp.

2. Totally k-dot-critical graphs with no critical vertices

The *Cartesian product* $G\Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G\Box H)$ whenever $x = y$ and $ab \in E(G)$, or $a = b$ and $xy \in E(H)$.

Burton and Sumner [1] showed following lemmas.

Lemma 2.1. *If G is any graph with $\gamma(G) = k \geqslant 2$, then G is dot-critical* (resp. *totally dot-critical*) *if and only if every two adjacent non-critical vertices* (resp. *any two non-critical vertices*) *belong to a common MDS.*

Lemma 2.2. *If G is a dot-critical graph* (resp. *totally dot-critical graph*) *and $N[v] \subseteq N[u]$, then $v \in G'$.*

Klavžar and Seifter [3] studied the domination number of $C_4\Box C_k$, and showed:

Lemma 2.3. $\gamma(C_4\Box C_k) = k$ $(k \geqslant 4)$.

We study the graphs $C_4\Box C_4$, $C_4\Box C_5$ and $C_3\Box C_6$. The following observation is straightforward to verify and we omit the details.

Observation 2.4. $C_4\Box C_4$ is a totally 4-dot-critical graph with $G' = \emptyset$, $C_4\Box C_5$ is not a totally 5-dot-critical with $G' = \emptyset$, and $C_3\Box C_6$ is a totally 5-dot-critical graph with $G' = \emptyset$.

Now, we consider the cases for $k \geqslant 6$.

Lemma 2.5. $\gamma((C_4\Box C_k).vu) = k - 1$ *for any u and v in* $V(C_4\Box C_k)$ $(k \geqslant 6)$.

Proof. Without loss of generality, we need only to consider the cases for $u = v_{0,0}$, $v = v_{i,j}$ $((i, j) \neq (0, 0))$.