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On the pebbling threshold of paths and the pebbling threshold spectrum

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Abstract

A configuration of pebbles on the vertices of a graph is solvable if one can place a pebble on any given root vertex via a sequence of pebbling steps. A function is a pebbling threshold for a sequence of graphs if a randomly chosen configuration of asymptotically more pebbles is almost surely solvable, while one of asymptotically fewer pebbles is almost surely not. In this paper we tighten the gap between the upper and lower bounds for the pebbling threshold for the sequence of paths in the multiset model. We also find the pebbling threshold for the sequence of paths in the binomial model. Finally, we show that the spectrum of pebbling thresholds for graph sequences in the multiset model spans the entire range from $n^{1/2}$ to n, answering a question of Czygrinow, Eaton, Hurlbert and Kayll. What the spectrum looks like above n remains unknown.

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1. Introduction

Let $G=(V, E)$ be a connected graph on n vertices and let D be a *configuration* of t unlabeled pebbles on V (formally D is multiset of t elements from V, with $D(v)$ the number of pebbles on vertex v). A *pebbling step* consists of removing two pebbles from a vertex v and placing one pebble on a neighbor of v. A configuration is called *r-solvable* if it is possible to move at least one pebble to vertex r by a sequence of pebbling steps. A configuration is called *solvable* if it is r-solvable for every vertex $r \in V$. The *pebbling number* of G is the smallest integer $\pi(G)$ such that every configuration of $t=\pi(G)$ pebbles on G is solvable. Pebbling problems have a rich history and we refer to [6] for a thorough discussion. Standard asymptotic notation will be used in the paper. For two functions $f=f(n)$ and $g=g(n)$, we write $f \ll g$ (or $f \in o(g)$) if f/g approaches zero as n approaches infinity, $f \in O(g)$ $(f \in \Omega(g))$ if there exist positive constants c, k such that $f < cg$ $(f > cg)$ whenever $n > k$. In addition, $f \in \Theta(g)$ when $f \in O(g)$ and $g \in O(f)$. We will also use $f \sim g$ if f/g approaches 1 as n approaches infinity. Finally to simplify the exposition we shall always assume, whenever needed, that our functions take integer values.

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We will be mainly interested in the following random model considered in [2]. A configuration D of t pebbles assigned to G is selected randomly and uniformly from all $\binom{n+t-1}{t}$ configurations. The problem to investigate, then, is to find what values of t, as functions of the number of vertices $n = n(G)$, make D almost surely solvable. More precisely, a function $t = t(n)$ is called a *threshold* of a graph sequence $\mathscr{G} = (G_1, \dots, G_n, \dots)$, where G_n has n vertices, if the following conditions hold as n tends to infinity:

1. for $t_1 \ll t$ the probability that a configuration of t_1 pebbles is solvable tends to zero, and
2. for $t_2 \gg t$ the probability that a configuration of t_2 pebbles is solvable tends to one.

We denote by $\tau_{\mathbf{M}}(\mathscr{G})$ the set of all threshold functions of $\mathscr{G}$ in the multiset model. It is not immediately clear, however, that $\tau_{\mathbf{M}}(\mathscr{G})$ is nonempty for all $\mathscr{G}$. Nonetheless it is proven to be the case in [1]. In this paper, we will study thresholds in the case when $\mathscr{G}$ is the family of paths. First let us note that the pebbling number of a path on n vertices is equal to 2^{n-1}. However, most of the configurations on t pebbles with t much smaller than 2^{n-1} will still be solvable and so not surprisingly the threshold of the family of paths is much smaller than 2^{n-1}. Let $\mathscr{P} = (P_n)_{n=1}^{\infty}$ be the sequence of paths. In [1] it is showed that

$$\tau_{\mathbf{M}}(\mathscr{P}) = \mathrm{O}(n2^{2\sqrt{\lg n}}) \tag{1}$$

and

$$\tau_{\mathbf{M}}(\mathscr{P}) = \Omega(n2^{c\sqrt{\lg n}}) \tag{2}$$

for any constant $c < 1/\sqrt{2}$. The upper bound (1) was improved by Godbole et al. [5], to

$$\tau_{\mathbf{M}}(\mathscr{P}) = \mathrm{O}(n2^{C\sqrt{\lg n}}) \tag{3}$$

for any constant $C > 1$. Our main result of the paper improves the lower bound from [1], showing a lower bound which almost matches the upper bound from [5].

Theorem 1. *Let $\mathscr{P} = (P_n)_{n=1}^{\infty}$ be the sequence of paths. For any $\delta > 0$, let $w = (1-\delta)\sqrt{\lg n}$. Then $\tau_{\mathbf{M}}(\mathscr{P}) = \Omega(n2^{w})$.*

Clearly the random pebbling model from [2] is only one of many that can be considered. In particular, if pebbles are distinguishable and each of them selects independently at random a vertex to be placed on then we obtain a completely different model, which we call the binomial model. We can define the threshold $\tau_{\mathbf{B}}(\mathscr{G})$ in this model in the same way that $\tau_{\mathbf{M}}(\mathscr{G})$ is defined for the multinomial model. Then it is easy to see that $\tau_{\mathbf{B}}(\mathscr{P}) = \mathrm{O}(n \ln n)$ (since the probability that every vertex contains a pebble tends to 1) but in fact the threshold is slightly smaller.

Theorem 2. *Let $\mathscr{P} = (P_n)_{n=1}^{\infty}$ be the sequence of paths. Then*

$$\tau_{\mathbf{B}}(\mathscr{P}) = \left(\frac{1}{2} + \mathrm{o}(1)\right) n \frac{\ln n}{\lg \ln n}.$$

It turns out that to prove Theorem 1 it is convenient to consider one more model, the geometric one. In this model each vertex on a path generates the number of pebbles that it contains according to the geometric distribution with $p = t/(t+n)$, where t is some function of n—that is, the probability that exactly C pebbles sit on a fixed vertex equals $p^C(1-p)$. Conveniently, the geometric model can be used to approximate the multinomial one from [2]. It is this observation that allows us to generalize the technique from [1] and prove a better lower bound.

The rest of the paper is organized as follows. We prove Theorem 1 in Section 2, and in Section 3 we show Theorem 2. Finally, Section 4 is devoted to investigating which functions $t = t(n)$ can be pebbling thresholds in the multiset model for some sequence of graphs. In particular, we verify the following conjecture posed in [2].

Conjecture 3. For every $\Omega(n^{1/2}) \ni t_1 \ll t_2 \in \mathrm{O}(n)$ there exists a graph sequence $\mathscr{G} = (G_1, \dots, G_n, \dots)$ such that $\tau_{\mathbf{M}}(\mathscr{G}) \subset \Omega(t_1) \cap \mathrm{O}(t_2)$.