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Note

Hoàng–Reed conjecture holds for tournaments

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Abstract

Hoàng–Reed conjecture asserts that every digraph D has a collection $\mathscr{C}$ of circuits $C_1, \ldots, C_{\delta^+}$, where δ^+ is the minimum outdegree of D, such that the circuits of $\mathscr{C}$ have a forest-like structure. Formally, $|V(C_i) \cap (V(C_1) \cup \cdots \cup V(C_{i-1}))| \leqslant 1$, for all $i = 2, \ldots, \delta^+$. We verify this conjecture for the class of tournaments.

Keywords: Tournament; Triangle structure

1. Introduction

One of the most celebrated problems concerning digraphs is the Caccetta–Häggkvist conjecture (see [1]) asserting that every digraph D on n vertices and with minimum outdegree n/k has a circuit of length at most k. Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng–Reed conjecture [3] is one of these.

A *circuit-tree* is either a singleton or consists of a set of circuits $C_1, \ldots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \cdots \cup V(C_{i-1}))| = 1$ for all $i = 2, \ldots, k$, where $V(C_j)$ is the set of vertices of C_j. A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique xy-directed path for every distinct vertices x and y. A vertex-disjoint union of circuit-trees is a *circuit-forest*. When all circuits have length three, we speak of a *triangle-tree*. For short, a k-circuit-forest is a circuit-forest consisting of k circuits.

Conjecture 1 (*Hoàng and Reed [3]*). Every digraph has a δ^+-circuit-forest.

This conjecture is not even known to be true for $\delta^+ = 3$. In the case $\delta^+ = 2$, Thomassen [6] proved that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng–Reed conjecture is that it would imply the Caccetta–Häggkvist conjecture, as the reader can

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easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament T, that is the 3-uniform hypergraph on vertex set V which edges are the 3-circuits of T.

Indeed, if a tournament T has a δ^+-circuit-forest, by the fact that every circuit contains a directed triangle, T also has a δ^+-triangle-forest. Observe that a δ^+-triangle-forest spans exactly $2\delta^+ + c$ vertices, where c is the number of components of the triangle-forest. When T is a regular tournament with outdegree δ^+, hence with $2\delta^+ + 1$ vertices, a δ^+-triangle-forest of T is necessarily a spanning δ^+-triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

Theorem 1. *Every tournament has a δ^+-triangle-tree.*

2. Components in bipartite graphs

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

Lemma 1. *Let $k \geqslant 1$ and let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be two sequences of positive reals. Let $A = \sum_{i=1}^k a_i$ and $B = \sum_{j=1}^k b_j$. If $\sum_{i=1}^k a_i b_i = (AB/2) + q$, where $q \geqslant 0$, then there is an i such that $a_i + b_i \geqslant ((A+B)/2) + \sqrt{2q}$.*

Proof. If $k = 1$, then the lemma follows immediately as $q = AB/2$ and $A + B \geqslant ((A+B)/2) + \sqrt{AB}$. So assume that $k > 1$. Without loss of generality, we may assume that $(a_1, b_1) \geqslant (a_2, b_2) \geqslant \cdots \geqslant (a_k, b_k)$ in the lexicographical order. Let r be the minimum value such that $b_r \geqslant b_i$ for all $i = 1, 2, \ldots, k$. Note that $a_1 \geqslant |A|/2$, since otherwise $\sum_{i=1}^k a_i b_i < \sum_{i=1}^k A b_i/2 = AB/2$. Analogously $b_r \geqslant |B|/2$. Define a' and b' so that $a_1 = A/2 + a'$ and $b_r = B/2 + b'$.

If $r \neq 1$, then the following holds:

$$\begin{aligned}\sum_{i=1}^k a_i b_i &\leqslant a_1 b_1 + \sum_{i=2}^k a_i b_r \\ &\leqslant a_1(B - b_r) + (A - a_1) b_r \\ &= \left(\frac{A}{2} + a'\right)\left(\frac{B}{2} - b'\right) + \left(\frac{A}{2} - a'\right)\left(\frac{B}{2} + b'\right) \\ &= \frac{AB}{2} - 2a'b' \\ &\leqslant \frac{AB}{2}.\end{aligned}$$

As $q \geqslant 0$, this implies we have equality everywhere above, which means that $b_1 = B - b_r$. As $B = b_1 + b_r$, we must have $k = 2$. As there was equality everywhere above we have $b' = 0$ or $a' = 0$ which implies that $a_1 = a_2 = A/2$ or $b_1 = b_2 = B/2$. In both cases we would have $r = 1$, a contradiction.

Suppose now that $r = 1$. Then

$$\frac{AB}{2} + q \leqslant a_1 b_1 + (A - a_1)(B - b_1) = \left(\frac{A}{2} + a'\right)\left(\frac{B}{2} + b'\right) + \left(\frac{A}{2} - a'\right)\left(\frac{B}{2} - b'\right).$$

This implies that $q \leqslant 2a'b'$. The minimum value of $a' + b'$ is obtained when $a' = b' = \sqrt{q/2}$. Therefore, the minimum value of $a_1 + b_1$ is $A/2 + B/2 + 2\sqrt{q/2}$. This completes the proof of the lemma. □

Corollary 1. *Let G be a bipartite graph with partite sets A and B. If $|E(G)| = (|A||B|/2) + q$, where $q \geqslant 0$, then there is a component in G of size at least $|V(G)|/2 + \sqrt{2q}$.*