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Discrete Mathematics 308 (2008) 3420–3426
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A characterization of $(2\gamma, \gamma_{\mathrm{p}})$-trees ☆

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Received 28 December 2005; received in revised form 29 April 2007; accepted 21 June 2007
Available online 2 August 2007

Abstract

Let $G=(V, E)$ be a graph. A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent with some vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a paired-dominating set of G if S dominates V and $\langle S\rangle$ contains at least one perfect matching. The paired-domination number of G, denoted by $\gamma_{\mathrm{p}}(G)$, is the minimum cardinality of a paired-dominating set of G. In this paper, we provide a constructive characterization of those trees for which the paired-domination number is twice the domination number.

Keywords: Domination; Paired-domination; Tree

1. Introduction

Let $G=(V, E)$ be a graph with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$ is $N(v)=\{u \in V | uv \in E\}$, the set of vertices adjacent to v. The closed neighborhood of v is $N[v]=N(v) \cup \{v\}$. For $S \subseteq V$, the open neighborhood of S is defined by $N(S)=\cup_{v\in S} N(v)$, and the closed neighborhood of S by $N[S]=N(S) \cup S$. The private neighborhood $\mathrm{PN}(v, S)$ of $v \in S$ is defined by $\mathrm{PN}(v, S)=N(v)-N[S-\{v\}]$. The private neighborhood $\mathrm{PN}(S', S)$ of $S' \subset S$ is defined by $\mathrm{PN}(S', S)=N(S')-N[S-S']$. The subgraph of G induced by the vertices in S is denoted by $\langle S\rangle$. For $X, Y \subseteq V$, if X dominates Y, we write $X \succ Y$, or $X \succ G$ if $Y=V$, or $X \succ y$ if $Y=\{y\}$.

A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent to some vertex in S. (That is, $N[S]=V$.) The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G (similar notation is used for the other domination parameters).

Let $G=(V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a paired-dominating set of G if S dominates V and $\langle S\rangle$ contains at least one perfect matching M. If an edge $uv \in M$, we say that u and v are paired in S. The paired-domination number of G, denoted by $\gamma_{\mathrm{p}}(G)$, is the minimum cardinality of a paired-dominating set of G. Paired-domination in graphs was introduced by Haynes and Slater [7]. Recall that a dominating set $S \subseteq V$ of G is a total dominating set if $\langle S\rangle$ contains no isolated vertices. The total domination number of G, denoted by $\gamma_{\mathrm{t}}(G)$, is the minimum cardinality of a total dominating set of G. Clearly, $\gamma_{\mathrm{t}}(G) \leqslant \gamma_{\mathrm{p}}(G)$ for every connected graph with order at least two. Total domination in graphs was introduced by Cockayne et al. [1]. The concept of domination in graphs, with its many variations, is well studied in graph theory (see [4,5]).

☆ The work was supported by NNSF of China (No. 10671191).
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doi:10.1016/j.disc.2007.06.034

An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters. For any two graph theoretic parameters λ and μ, G is called a (λ, μ)-graph if $\lambda(G) = \mu(G)$. The class of (γ, i)-trees, that is, trees with equal domination and independent domination numbers was characterized in [2]. In [3], the authors provided a constructive characterization of trees with equal independent domination and restrained domination numbers, and a constructive characterization of trees with equal independent domination and weak domination numbers is also given. In [9], the authors characterized those trees with equal domination and paired-domination numbers. In [8], those trees with equal domination and total domination numbers were characterized. In [6], the authors provided a constructive characterization of the trees T for which (1) $\gamma(T) \equiv i(T)$; (2) $\gamma(T) \equiv \gamma_t(T)$; and (3) $\gamma(T) \equiv \gamma_p(T)$.

Clearly, if G has a paired-dominating set, then $\gamma_p(G)$ is even. For the domination and paired-domination numbers, we have

Fact 1 (*Haynes and Slater [7]*). *Let G be a graph without isolated vertices. Then, G has a paired-dominating set, and* $\gamma_p(G) \leqslant 2\gamma(G)$.

In this paper, we give a constructive characterization of trees for which the paired-domination number is twice the domination number.

2. Main result

Let $T = (V, E)$ be a tree with vertex set V and edge set E. A vertex of T is said to be remote if it is adjacent to a leaf. The set of leaves of T is denoted by $L(T)$. In this paper, we use T_v to denote the subtree of $T - uv$ containing v for $uv \in E(T)$. P_ℓ represents a path with l vertices. $|T|$ denotes the order of a tree T.

We begin with a proposition about the paired-dominating set of a tree T.

Proposition 2. *If S is a γ_p-set of a tree T, then $\langle S \rangle$ contains a unique perfect matching.*

Proof. Let H be a component of $\langle S \rangle$, then H has a perfect matching. So $|H|$ is even. To prove that $\langle S \rangle$ contains a unique perfect matching, it is enough to show that H has a unique perfect matching. We prove H has a unique perfect matching by induction on $2n$, the order of H. If $n = 1$, then $H \cong K_2$, the result is clearly true. Let $n > 1$ and assume that the result is true for every tree H' of order $< 2n$, where H' is a tree containing a perfect matching. Let H be a tree of order $2n$ containing a perfect matching M. Let u be a leaf of H and v be the remote vertex such that $uv \in E(H)$. Then $uv \in M$ and u is the unique leaf adjacent to v in H. Let $H_1, H_2, \ldots, H_k$ be the components of $H - \{u, v\}$. Then every H_i has a perfect matching and $|H_i| < 2n$. By inductive hypothesis, H_i has a unique perfect matching M_i. So, $M = (\cup_{i=1}^{k} M_k) \cup \{u, v\}$ is the unique perfect matching of H. The result follows. □

Let S be a paired-dominating set of a tree T. By Proposition 2, S has a unique perfect matching M. So, for any vertex $v \in S$, the paired vertex of v is unique. We denote the unique paired vertex of $v \in S$ by $\bar{v}$.

To state the characterization of $(2\gamma, \gamma_p)$-trees, we introduce three types of operations.

Type-1 *operation*: Attach a path P_1 to a vertex v of a tree T, where v is in a γ-set of T and $v \notin L(T)$. (As shown in Fig. 1(a).)

Type-2 *operation*: Attach a path P_2 to a vertex v of a tree T, where v is a vertex such that for every γ_p-set S of T containing v, $\mathrm{PN}(v, S) = \emptyset$ and $\mathrm{PN}(\{v, \bar{v}\}, S) \neq \emptyset$. (As shown in Fig. 1(b).)

Type-3 *operation*: Attach a path P_3 to a vertex v of a tree T, where either v is a vertex of a γ-set of T such that $v \notin L(T)$ and, for every γ_p-set S of T, $\mathrm{PN}(\{v, \bar{v}\}, S) \neq \emptyset$ if $\bar{v} \notin L(T)$, or v is a vertex such that for every γ_p-set S of T containing v, $\bar{v} \notin N(S - \{v, \bar{v}\})$ if $\mathrm{PN}(\{v, \bar{v}\}, S) = \emptyset$. (As shown in Fig. 1(c).)

Let $\mathscr{I}_p$ be the family of trees for which the paired-domination number is twice the domination number, that is

$$\mathscr{I}_p = \{T : \gamma_p(T) = 2\gamma(T)\}.$$

We define the family $\mathscr{F}_p$ as:

$\mathscr{F}_p = \{T : T$ is obtained from P_3 by a finite sequence of operations of Type-1, Type-2 or Type-3$\}$.

We shall prove that