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Discrete Mathematics 308 (2008) 3073–3081
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A combinatorial approach to doubly transitive permutation groups

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Received 27 August 2004; accepted 9 August 2007
Available online 3 October 2007

Abstract

Let G be a doubly but not triply transitive group on a set X. We give an algorithm to construct the orbits of G acting on $X \times X \times X$ by combining those of its stabilizer H of a point of X If the group H is given first, we compute the orbits of its transitive extension G, if it exists. We apply our algorithm to $G = \mathrm{PSL}(m, q)$ and $\mathrm{Sp}(2m, 2)$, $m \geqslant 3$, successfully. We go forward to compute the transitive extension of G itself. In our construction we use a superscheme defined by the orbits of H on $X \times X \times X$ and do not use group elements.

MSC: 20B20; 20B40; 05E30

Keywords: Multiply transitive finite groups; Association schemes; Superschemes; Computational method

1. Introduction

Known doubly transitive permutation groups are well studied and the classification of all doubly transitive groups has been done by applying the classification of the finite simple groups. Readers may refer to [8]. This may mean that it is not still sufficient to study doubly transitive groups as permutation groups. Typical arguments on doubly transitive groups are seen in the book [5]. In the present paper we will give a combinatorial approach to doubly transitive groups. Let G be a doubly transitive permutation group on a set X and let G_α be the stabilizer of a point $\alpha \in X$ in G. We use a combinatorial structure called a superscheme defined by G_α and try to construct the superscheme defined by the doubly transitive group G. We do not use any group elements in this construction. Then the automorphism group of the constructed superscheme can be expected to be closely related to the group G. However, we will not compute the automorphism group generally, since we can usually guess what a doubly transitive group it is. In turn, if a superscheme defined by a transitive group G_α is given first, our algorithm will construct the superscheme defined by the expected transitive extension of G_α. If no superscheme is constructed or none of the constructed superschemes have automorphism groups transitive on X, then we can conclude that there is no transitive extension of G_α.

There are some references on transitive extensions [4,1,12], especially showing the non-existence of transitive extensions of some doubly transitive groups themselves. Noda [12] considers some permutation structure of G_α acting on X^3, the set of triples of X. In the present paper we consider the orbits of G_α acting on X^3, which define a superscheme, and by our algorithm we will obtain the orbits of G on X^3, if G_α has a transitive extension G.

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doi:10.1016/j.disc.2007.08.032

We will show how our algorithm works in the case of projective special linear groups $\mathrm{PSL}(m, q)$, $m \geqslant 3$. Our algorithm also works successfully in the case of symplectic groups $\mathrm{Sp}(2m, 2)$, $m \geqslant 3$, over GF(2) acting on the cosets by $\mathrm{O}^+(2m, 2)$ and $\mathrm{O}^-(2m, 2)$. In PSL cases the automorphism groups of the obtained superschemes are expected to be $P\Gamma L$. We also compute the transitive extensions of these groups themselves. Our algorithm shows that they do not have transitive extensions except $\mathrm{PSL}(m, 2)$ and $\mathrm{PSL}(3, 4)$, which are known to give $\mathrm{AGL}(m, 2)$ and the Mathieu group M_{22} of degree 22, and, so on, we can consider the extensions of the obtained groups. For the transitive extensions of groups H such that $\mathrm{PSL}(m, q) \subseteq H \subseteq P\Gamma L(m, q)$, readers may refer to Lemma 3. Computer experiments suggest that our algorithm can be applied to any doubly or more transitive groups. We only note that Higman–Sims group of degree 176 and Conway 3 group of degree 276 can be computed.

Association schemes are often used to study permutation groups and superschemes are a kind of generalization of association schemes. A doubly transitive group of degree n defines a trivial association scheme which is the same one given by the symmetric group of degree n. So it is worthless to consider the association schemes given by doubly transitive groups. If a group G is doubly but not triply transitive, the stabilizer of a point in G defines a non-trivial association scheme. But as it is mentioned in [11], this association scheme is not sufficient to construct the group G. We will repeat about this fact briefly in the present paper. So we will use superschemes to construct doubly transitive groups from their stabilizers of a point. Superschemes are introduced in [7,13]. We will follow a slightly different definition of superschemes given in [6].

We note that we used `GAP` [3] for our computer experiments. Our programs are written based on those in [9,10].

2. Association schemes and superschemes

Let $X = \{x_1, x_2, \ldots, x_s\}$ be the set of vertices. Then an association scheme $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ and a superscheme (X, Π) are defined as follows.

Definition. $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ is an *association scheme* if and only if

(1) $R_0 = \{(x, x) | x \in X\}$,
(2) $\{R_0, R_1, \ldots, R_d\}$ is a partition of $X \times X$,
(3) for all R_i there exists $R_{i'}$ such that $R_{i'} = \{(y, x) | (x, y) \in R_i\}$,
(4) for all R_i, R_j, R_k and for all $(x, y) \in R_k$, there exists a constant number p_{ijk} such that

$$p_{ijk} = |\{z \in X | (x, z) \in R_i, (z, y) \in R_j, (x, y) \in R_k\}|.$$

Definition. (X, Π) is a *superscheme* if and only if

(1) $\Pi = \{\Pi^1, \Pi^2, \ldots, \Pi^t\}$ for some $t \geqslant 2$, Π^l is a partition of X^l for $1 \leqslant l \leqslant t$,
(2) let $\sigma((y_1, y_2, \ldots, y_l)) = (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(l)})$ for $\sigma \in \mathrm{Sym}(l)$, let $\Pi^l = \{R^l_0, R^l_1, \ldots, R^l_{d_l}\}$, $1 \leqslant l \leqslant t$, then $\sigma(R^l_k) \in \Pi^l$ for all R^l_k and all $\sigma \in \mathrm{Sym}(l)$,
(3) let a projection $\pi^l_j : X^l \to X^{l-1}$ be defined by

$$\pi^l_j((y_1, y_2, \ldots, y_l)) = (y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_l),$$

then $\pi^l_j(R^l_k) \in \Pi^{l-1}$ for all $R^l_k \in \Pi^l$, $2 \leqslant l \leqslant t$ and $1 \leqslant j \leqslant l$,
(4) for all R^l_k, $2 \leqslant l \leqslant t$, for all $\boldsymbol{y} = (y_1, y_2, \ldots, y_{l-1}) \in \pi^l_j(R^l_k)$, $1 \leqslant j \leqslant l$, there exists a constant number $p^l_{k,j}$ such that $p^l_{k,j} = |(\pi^l_j)^{-1}(\boldsymbol{y}) \cap R^l_k|$. In particular, $p^l_{k,j} = |R^l_k| / |\pi^l_j(R^l_k)|$.

For more properties of association schemes, readers may refer to [2]. The properties (2) and (4) of a superscheme are called *symmetric* and *regular* in [6], respectively. Referring to the number t in the definition of a superscheme, we simply call a t-superscheme. Each R^l_i is called a *relation*.