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Unimodular modules

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Abstract

The present publication is mainly a survey paper on the author's contributions on the relations between graph theory and linear algebra. A system of axioms introduced by Ghouila-Houri allows one to generalize to an arbitrary Abelian group the notion of interval in a linearly ordered group and to state a theorem that generalizes two due to A.J. Hoffman. The first is on the feasibility of a system of inequalities and the other is Hoffman's circulation theorem reported in the first Berge's book on graph theory. Our point of view permitted us to prove classical results of linear programming in the more general setting of linearly ordered groups and rings. It also shed a new light on convex programming.

Keywords: Graph theory; Linear algebra; Abelian group; Ring; Modules; Generator; Linearly ordered group; Linearly ordered ring; Convexity; Polyhedron; Matroid

1. Introduction

1.1. Motivations

We refer here to Berge's book [3]. All results in the present paper have their origin in Section 15 of that book, translated into English from [1], and in Berge's subsequent paper [2]. The aim of this paper, which is a rewriting of part of [14], is to gather together many of the results known already in 1965 and to set them in an algebraic context.

Rado [37] was the first to use properties of linear inequalities to prove the König–Hall Theorem on the existence of a matching in a bipartite graph. Hoffman [26,25] then showed that the König–Hall Theorem and its derivatives were but a reflection of the total unimodularity of incidence matrices of vertices versus arcs of directed graphs. More precisely he gave a combinatorial result on totally unimodular matrices (a^i_j) over $\mathbb{Z}$ with positive entries which generalizes König–Hall Theorem. His proof is based upon linear inequalities. Previously, Tutte [38–40] and Heller [24,23] had

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[1] J'ai rencontré Claude Berge en 1959 alors que je travaillais sous la direction de Paul Gillis au Centre d'Etudes et de Recherche Opérationnelle, que celui-ci venait de créer Bruxelles. Claude Berge accompagnait d'autres personnalités scientifiques parmi lesquelles Robert Fortet, Jacques Lesourne et Germain Kreweras. Claude Berge, par ses exposés sur les graphes, a suscité mon enthousiasme pour la théorie des graphes, et de là pour les mathématiques du discret. Il fut à l'origine de l'orientation et du développement de mes travaux de recherche.

I met Claude Berge in 1959 when I was working under the direction of Paul Gillis at the Centre dEtudes et de Recherche Oprationnelle that he had just created in Brussels. Claude Berge was one of a number of scientists there, including Robert Fortet, Jacques Lesourne and Germain Kreweras. Through his talks on graphs, Claude Berge aroused my enthusiasm for graph theory, and for discrete mathematics in general. The direction my research took, and its subsequent development, originated with him.

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shown how useful considering minimal sets of columns of totally unimodular matrices over $\mathbb{Z}$ could be; these minimal subsets generalize the notions of elementary cycles and elementary cocycles in graphs. In [12] it was shown how both those points of view could be reconciled by defining unimodular modules in $\mathbb{Z}^m$ where vectors with minimal support play an essential part.

A system of axioms introduced by Ghouila-Houri [20] allows one to generalize to an arbitrary Abelian group the notion of interval in a linearly ordered group and to state a theorem that generalizes two due to Hoffman. The first [25] is on the feasibility of a system of inequalities and the other is Hoffman's circulation Theorem reported in [1]. It states the conditions for the existence of a flow with components in prescribed intervals of integers.

The latter Theorem of Hoffman unified a significant number of results in graph theory, including the theorems of Gale, Ford and Fulkerson and Menger. By this very fact they are a reflection of properties of unimodular modules. The present approach would not have been possible without the results of Ghouila-Houri [20], Minty [35] and Berge [2]. Properties of linear inequalities that we consider are collected in a book edited by Kuhn and Tucker [29]. It turns out that many of them are corollaries of Theorem 3.1. Later, our point of view permitted us to prove classical results of linear programming in the more general setting of linearly ordered groups and rings. It also shed a new light on convex programming [13]. Let us finally mention that in 1963 Alan Hoffman communicated his enthusiasm to the author, in particular regarding the work on linear inequalities, while the author was visiting IBM Watson Research Center [16].

1.2. Foreword

A word of explanation is perhaps in order. Our aim here is not to update our earlier paper "Modules unimodulaires", but rather to take advantage of this English version of [14] to rectify several deficiencies in the original paper. The easy Proposition 2.1 was not stated in [14]. It will turn out to be essential to the proof of Theorem 3.1. All proofs of the results stated in Sections 5.2 and 5.3 rely on Corollary 2.1 which was not stated in [14].

Theorem 5.7 shows how the well-known duality theorem of linear programming can be extended to polynomial rings over linearly ordered fields. Next, the Dantzig simplex method is also extended to polynomial rings over linearly ordered fields. This was essentially contained in the exposition of [13] but it was not clearly stated. A number of the proofs there are not given here since they can easily be established by the reader.

Let us point out that the present version is intended to take into account further results published in 1974 [15], for which the paper "Modules unimodulaires" [14] was a natural precursor.

For instance Farkás' Lemma [19], which is not considered in [14] has a more general setting in [15] (Section 3.2, see Theorem 7) where unimodular modules are generalized to stable modules. Also, the proof of Theorem 11 in section 3 of [15] is essentially that of Theorem 5.5. Briefly, in a stable module, the set of units in a ring R needed to define unimodular modules is extended to any subset $S \subset R$ such that $SS \subset S$.

Several proofs in [14] lead to algorithms. For instance the generators of the polar of a cone can be constructed by using the constructive proof of the partition lemma (Lemma 5.1), which generalizes Minty's lemma [35]. Remark 5.7 also points out that an algorithm exists to actually express any vector in a polyhedron as a linear combination of some of the extreme points of that polyhedron. Such constructive arguments already appeared in the proof of the generalized Farkás' Lemma (Theorem 7) as well as in that of Theorem 11 in Section 3 of [15]. Let us also point out that the proof of Theorem 4.1 is constructive. Michel Las Vergnas suggested an oriented matroid version of that result. It is still to be found.

We thus emphasize that the present paper is not intended to update the paper [14] and in particular to quote all subsequent results on oriented matroids (introduced independently by Las Vergnas [31], Bland [7] and Lawrence [33]). An extensive account on oriented matroids can be found in [6] beside the original papers by Las Vergnas [32], and Bland [8,9].

1.3. A first insight

Basic examples of unimodular modules over the ring $\mathbb{Z}$ of integers are the modules spanned by the rows of a cyclomatic matrix or a cocyclomatic matrix (see [3], [Section 15, Theorem 5 (Poincaré–Veblen–Alexander)]). Theorem 5.1 states: *Let L be a unimodular module in G^m. Every semi-positive vector x of L is positive linear combination of semi-positive generators with supports contained in that of x.* By the theorem of Poincaré–Veblen–Alexander, the module over $\mathbb{Z}$ spanned by the rows of the incidence matrix of a graph is unimodular and its orthogonal module, denoted by L, is the