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On affine designs and Hadamard designs with line spreads[☆]

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Dedicated to Jennifer Seberry on her 60th birthday

Abstract

Rahilly [On the line structure of designs, Discrete Math. 92 (1991) 291–303] described a construction that relates any Hadamard design H on $4^m - 1$ points with a line spread to an affine design having the same parameters as the classical design of points and hyperplanes in $AG(m, 4)$. Here it is proved that the affine design is the classical design of points and hyperplanes in $AG(m, 4)$ if, and only if, H is the classical design of points and hyperplanes in $PG(2m - 1, 2)$ and the line spread is of a special type. Computational results about line spreads in $PG(5, 2)$ are given. One of the affine designs obtained has the same 2-rank as the design of points and planes in $AG(3, 4)$, and provides a counter-example to a conjecture of Hamada [On the p-rank of the incidence matrix of a balanced or partially balanced incomplete block design and its applications to error-correcting codes, Hiroshima Math. J. 3 (1973) 153–226].

Keywords: Affine design; Affine geometry; Hadamard matrix; Hamada's conjecture

1. Introduction

The connection between Hadamard matrices and symmetric or affine designs is well-known, see [12] for example. In this paper, we describe two constructions, based on one of Rahilly [10], that relate affine 2-designs of class number 4 with symmetric Hadamard 2-designs possessing spreads of lines where each line has size 3. In Section 2, we show that the affine design is the classical design of points and hyperplanes in the affine geometry $AG(m, 4)$ of dimension m over the field of four elements if, and only if, the Hadamard design is the classical design of points and hyperplanes in the projective geometry $PG(2m - 1, 2)$ of dimension $2m - 1$ over the field of two elements and the line spread is of a special type which we call *normal* and define in 2.5 below. In Section 3, we give an indication of the variety of affine designs produced by this construction using line spreads from the projective geometry $PG(5, 2)$ by summarizing computational results. In particular, we establish the falsity of Hamada's conjecture that, among the 2-designs with the same parameters as the 2-design of points and t-subspaces of a projective or affine geometry over a field of characteristic p, the designs whose incidence matrices are of minimum p-rank are isomorphic to the given design of points and t-subspaces of a projective or affine geometry, by exhibiting a non-geometric affine 2-(64,16,5) design, whose incidence matrix has 2-rank 16. Although it has not yet been established that 16 is the minimum 2-rank of the incidence matrices of 2-(64,16,5) designs, any 2-(64,16,5) designs of 2-rank less than 16 which might be discovered in the future will necessarily be non-geometric.

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The basic design theory needed for this paper may be found, for example, in [2,4,11]. We give an outline here.

Let $\Pi = (\mathscr{P}, \mathscr{B}, I)$ be a design with point set $\mathscr{P}$, block set $\mathscr{B}$ and incidence relation $I \subseteq \mathscr{P} \times \mathscr{B}$. Where convenient, as is customary, we may identify a block with the subset of points incident with it, and regard incidence as set-theoretic inclusion. Π is a t-(v, k, λ) design if $\mathscr{P}$ and $\mathscr{B}$ are finite, $|\mathscr{P}| = v$ and $|B| = k$ for all $B \in \mathscr{B}$ and any t-subset of $\mathscr{P}$ is contained in λ blocks. A design is *symmetric* if $|\mathscr{P}| = |\mathscr{B}|$. A t-(v, k, λ) design is *resolvable* if $\mathscr{B}$ has a partition, called a *parallelism*, into *parallel classes* of blocks such that two distinct blocks in the same parallel class are always disjoint and every point belongs to exactly one block from each parallel class. If, further, any two non-parallel blocks (i.e. blocks from different parallel classes) meet in a constant number $\mu > 0$ of points, then Π is *affine resolvable* or simply, *affine*. It is easy to see that each parallel class consists of $m = v/k$ blocks, where we call m the *class number* of the affine design, and $\mu = k/m$. From the definition, it follows that the parallelism in an affine design is unique.

The dual design Π^* of a design $\Pi = (\mathscr{P}, \mathscr{B}, I)$ is defined to be the design $\Pi^* = (\mathscr{B}, \mathscr{P}, I^*)$, where $(x, y) \in I$ if and only if $(y, x) \in I^*$. The *line* joining two distinct points P and Q in a t-design is the intersection of all blocks which contain both P and Q. If $t \geqslant 2$, the maximum size of a line is $(v-1)/k + 1$ and a line has this maximal size if, and only if, every block which does not contain it meets it in exactly one point. The set of blocks that contain the intersection of two distinct given blocks in Π forms a line in the dual design Π^*.

The parameters of a symmetric 2-(v, k, λ) design satisfy the equation $\lambda(v-1) = k(k-1)$. A symmetric 2-design is said to be *Hadamard* if $v = 2k + 1$. It is well-known that a Hadamard design exists if, and only if, a Hadamard matrix of order $2k+2$ exists or, equivalently, a 3-$(2k+2, k+1, \frac{1}{2}(k-1))$ design, which is necessarily affine, exists. The size of a line in a Hadamard 2-design is at most $(v-1)/k + 1 = 3$ since $v = 2k+1$. So, a line of size 3 has maximum size and any block either contains it or meets it in exactly one point.

A set $\mathscr{L}$ of non-empty point subsets of a design is a *spread* if it partitions the point set of the design. In a resolvable design, a parallel class of blocks is a spread of blocks.

2. Affine designs and spreads in symmetric designs

Rahilly [10] established a connection between affine 2-designs with class number 4 (the size of a parallel class) and Hadamard 2-designs with line spreads. This construction is generalized in Al-Kenani and Mavron [1]. Here, we present Rahilly's construction in a different but simpler and more transparent form that is suitable for the exposition of the results off this paper.

Construction 2.1. Let Γ be an affine 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu-1))$ design, where $\mu \equiv 1 \pmod 3$. Define a design Π as follows.

Choose any point w of Γ. The points of Π are all the points of Γ except w. To define a general block of Π, consider any parallel class $\mathscr{C}$. Then $\mathscr{C}$ has four blocks. Let B_0 be the block of $\mathscr{C}$ on w. For any $B \in \mathscr{C}$ with $B \neq B_0$, we define $B \cup B_0 - \{w\}$ to be a block of Π.

It is not difficult to verify that Π is a symmetric 2-$(16\mu-1, 8\mu-1, 4\mu-1)$ design and that, for any parallel class $\mathscr{C}$, the three blocks $B \cup B_0 - \{w\}$, with $B \in \mathscr{C}$ and $B \neq B_0$, form a line in the dual Π^* of Π. The set of all such lines is a spread of lines, each of size 3, in Π^*.

Construction 2.2. Let $\Pi = (\mathscr{P}, \mathscr{B}, I)$ be a symmetric 2-$(16\mu-1, 8\mu-1, 4\mu-1)$ design whose dual Π^* has a spread $\mathscr{L}$ of lines of size 3, that is, a set of lines of Π^* which partitions $\mathscr{B}$. Define an incidence structure Γ as follows.

The point set of Γ is $\mathscr{P} \cup \{w\}$ where w is a new point. The block set of Γ is $\mathscr{B} \cup \mathscr{L}$. We define the incidence relation I_Γ for Γ in two parts. Firstly, $w I_\Gamma L$ for all $L \in \mathscr{L}$. Secondly, let $P \in \mathscr{P}$ and $L \in \mathscr{L}$. If P is on exactly one (say B) of the three blocks of L in Π, then $P I_\Gamma B$. If P is on all three blocks of L in Π, then $P I_\Gamma L$.

It is routine to verify that Γ is an affine 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu-1))$ design. A typical parallel class consists of L, B_1, B_2, B_3, where $L \in \mathscr{L}$ and the B_i are the three blocks of L in Π.

The verifications for both of the above constructions may be found in Al-Kenani and Mavron [1] in a more general setting.

The constructions are, in an obvious sense, inverses of one another. However, it should be noted that the choice of spread in the second construction is important. Different choices of spread may result in non-isomorphic designs (see Section 3).