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Discrete Mathematics 308 (2008) 2896–2900
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On the range of influences in back-circulant Latin squares

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Received 15 September 2003; received in revised form 27 February 2004; accepted 9 March 2004
Available online 8 June 2007

Abstract

In this note we obtain a large lower bound for the index of a certain critical set in the back-circulant Latin squares of odd order. This resolves in the negative a conjecture of Fitina, Seberry and Chaudhry [Back-circulant Latin square and the influence of a set, Austral. J. Combin. 20 (1999) 163–180].
Published by Elsevier B.V.

Keywords: Latin square; Back-circulant Latin square; Index of critical set

1. Introduction

A number of authors have studied the properties of critical sets in various kinds of combinatorial designs. The paper [2] examines a particular critical set in the back-circulant Latin square. That paper discusses two concepts: the *influence* of an element of a critical set and the *index* of the critical set. The index of a given critical set is simply the number of distinct values taken by the influences of the elements in that set. In [2] the authors conjecture that the index of the particular critical set in the back-circulant Latin square of order $2m+1$ is $m-1$.

The conjecture is correct for $m \leqslant 7$, but an easy computer program reveals that the index for $m=8$ is 8 instead of 7. Therefore the conjecture is false. As it happens, in the present case, the index depends on the distribution of the integers which can be written in the form a^2+3b^2 where a and b are integers. Indeed, using known facts about the number of integers less than x which may be written in the above form, we are able to bound the index. The growth of the index is almost quadratic in m. Therefore the conjecture is in fact far from true.

2. Preliminaries

A *Latin square L* of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column. For convenience, a Latin square will sometimes be represented as a set of ordered triples $(i, j; k)$, which is read to mean that element k occurs in cell (i, j) of the Latin square L. A *partial Latin square P* of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs at most once in each row and column. Thus P may contain a number of empty cells. We sometimes denote P

[1] Research supported by Australian Research Council Grant DP0344078.
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0012-365X/$ - see front matter Published by Elsevier B.V.
doi:10.1016/j.disc.2004.03.024

by the set $\{(i,\ j;\ k)|i,\ j,\ k \in N\}$. A *critical set* in a Latin square L (of order n) is a set $\mathcal{C} = \{(i,\ j;\ k) \mid i,\ j, k \in N\}$, such that

(1) L is the only Latin square of order n which has element k in cell $(i,\ j)$ for each $(i,\ j;\ k) \in \mathcal{C}$; and
(2) no proper subset of $\mathcal{C}$ satisfies (1).

A *uniquely completable set* in a Latin square L of order n is a partial Latin square in L which satisfies condition (1) above.

Definition 1. In the process of completing the uniquely completable set U to the Latin square L of order n which it characterizes, we say that adjunction of a triple $t = (r, c; s)$ is *forced* (see [3]) in the process of completion of a set T of triples ($|\ T\ | < n^2$, $U \subseteq T \subset L$) to the complete set of triples which represents L, if either

(i) $\forall r' \neq r, \exists z \neq c$ such that $(r', z; s) \in T$ or $\exists z \neq s$ such that $(r', c; z) \in T$, or
(ii) $\forall c' \neq c, \exists z \neq r$ such that $(z, c'; s) \in T$ or $\exists z \neq s$ such that $(r, c'; z) \in T$, or
(iii) $\forall s' \neq s, \exists z \neq r$ such that $(z, c; s') \in T$ or $\exists z \neq c$ such that $(r, z; s') \in T$.

Put into words, condition (i) says that we must place s in the (r, c)th cell because for each unoccupied cell (r, c') in row r the number s already appears in column c'.

Let B_n denote the back-circulant Latin square of order n, based on the addition table for $\mathbb{Z}_n$. For all positive integers n,

$$B_n = \{(i,\ j;\ i + j \ (\mathrm{mod}\, n))|0 \leqslant i,\ j \leqslant n - 1\}.$$

It is well known (see [1]) that the following set

$$\{(i,\ j;\ i + j \ (\mathrm{mod}\, n))|(n + 1)/2 \leqslant i \leqslant n - 1 \text{ and } (3n - 1)/2 - i \leqslant j \leqslant n - 1\}$$
$$\cup \{(i,\ j;\ i + j \ (\mathrm{mod}\, n))|0 \leqslant i \leqslant (n - 3)/2 \text{ and } 0 \leqslant j \leqslant (n - 3)/2 - i\}$$

is a critical set in B_n for n odd. Let C_n denote this set. Note that C_n is the union of two equal triangles which we call the *upper triangle* and the *lower triangle*. Moreover, $|C_n| = (n^2 - 1)/4$. Fig. 1 shows the critical set C_{17}.

Definition 2. Let C be a critical set of order n in a Latin square L. For $x \in C$ we define the *nest* of x, denoted $\mathcal{N}(x)$, to be the union of $C \backslash \{x\}$ and the set of triples that can be forced (see Definition 1) when x is deleted from C. Define the *influence set* of x, denoted $\mathcal{I}(x)$, to be the set of cells $\{(i,\ j) \mid (i,\ j;\ k) \in (L \backslash \mathcal{N}(x))$ for some $k\}$. The number $|\mathcal{I}(x)|$ is called the *influence* of x, and denoted by $\theta(x)$.

Consider the elements (1, 2; 3) and (1, 1; 2) in critical set C_9. Fig. 2 illustrates some of these definitions.

0	1	2	3	4	5	6	7									
1	2	3	4	5	6	7										
2	3	4	5	6	7											
3	4	5	6	7												
4	5	6	7													
5	6	7														
6	7															
7																
																8
															8	9
														8	9	10
													8	9	10	11
												8	9	10	11	12
											8	9	10	11	12	13
										8	9	10	11	12	13	14
									8	9	10	11	12	13	14	15

Fig. 1. Critical set C_{17}.