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Weakly submodular rank functions, supermatroids, and the flat lattice of a distributive supermatroid

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In Memoriam Gian-Carlo Rota

Abstract

Distributive supermatroids generalize matroids to partially ordered sets. Completing earlier work of Barnabei, Nicoletti and Pezzoli we characterize the lattice of flats of a distributive supermatroid. For the prominent special case of a polymatroid the description of the flat lattice is particularly simple. Large portions of the proofs reduce to properties of weakly submodular rank functions. The latter are also investigated for their own sake, and some new results on general supermatroids are derived.

Keywords: Distributive supermatroid; Submodular function; Meet subsemilattice; Rank function; Greedoid; Selector; Supermatroid; Polymatroid; Semimodular lattice; Distributive lattice; Join-prime irreducible element; Independent order ideal; Unit increase; Closure space; Exchange property

1. Introduction

The correspondence between closure operators on a lattice $\mathscr{D}$ and $\wedge$-subsemilattices of $\mathscr{D}$ is well known. In Section 2 we review from [18] the correspondence between $\wedge$-subsemilattices and certain *weakly submodular rank functions* $\mathscr{D} \to \mathbb{N}$, and indicate how the applicability of weakly submodular rank functions (WSRFs) extends beyond the present article. In Section 3, starting with matroids, we introduce the more general class of Faigle-WSRFs and then the intermediate class of distributive supermatroids (DSMs). As is the case for matroids, each Faigle-WSRF admits a "simple" Faigle-WSRF with an isomorphic flat lattice. The corresponding fact for a DSM is less obvious and is established in Section 4. This is the basis for Section 5, where the flat lattice of a DSM is characterized. Clearly, the flat lattices of DSMs are more general than the geometric lattices linked to matroids, but more specific than the arbitrary upper semimodular lattices linked to Faigle-WSRFs. For DSMs both sides of the characterization are difficult: to (i) determine what kind of lattices arise, and (ii) to argue that any abstract such lattice stems from a suitable DSM.

In Section 6 we explain in quite a bit of detail how DSMs also fit the framework of selectors, greedoids, and of course (general) supermatroids. Some novelties concerning the latter, i.e. (15), (16), (19), will be established along the way. Sections 2 and 6 are likely the ones of broadest interest. This justifies the order of terms in the title, despite the fact that characterizing the flat lattice of a DSM constitutes the article's lion share. Without further mention, all sets and structures in this article are *finite*.

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2. The equivalence of ∧-subsemilattices and weak submodularity

We assume a basic familiarity with lattice theory. Since submodular functions and semimodular lattices occur frequently, the book of Stern [13] is particularly recommended as a reference for terms not fully defined and for additional background. Besides [13], we will also refer to [18], which in fact was triggered by previous drafts of the present article. Eventually enough results accumulated that held in a framework more general than DSMs, and which were sieved into [18]. Section 2 summarizes the key findings of [18], and adds Theorem 2 which was "missed" in [18].

Recall that a function cl: $\mathscr{D} \to \mathscr{D}$ is a *closure operator* if it is monotone, idempotent, and extensive in the sense that $A \leqslant \mathrm{cl}(A)$ for all A in $\mathscr{D}$. If $A = \mathrm{cl}(A)$, then A is *closed*. A $\wedge$-*subsemilattice* is a subset $\mathscr{L}$ of $\mathscr{D}$ such that $X \wedge Y \in \mathscr{L}$ for all $X, Y \in \mathscr{L}$. Here we also postulate $1_{\mathscr{D}} \in \mathscr{L}$. If $\mathscr{D} = \mathscr{B}(E)$ happens to be a Boolean lattice, i.e. the powerset of E, then the pair (E, cl) is often called a *closure space*. Generally, each closure operator cl: $\mathscr{D} \to \mathscr{D}$ yields the $\wedge$-subsemilattice $\mathscr{L} = \mathscr{L}[\mathrm{cl}]$ defined by

$$\mathscr{L} := \{\mathrm{cl}(A) \colon A \in \mathscr{D}\}.$$

On its own the set $\mathscr{L}$, partially ordered by $\leqslant$, is a lattice with meet $\triangle$ and join $\triangledown$ given by

$$X \triangle Y = X \wedge Y \quad \text{and} \quad X \triangledown Y = \mathrm{cl}(X \vee Y).$$

Conversely, each $\wedge$-subsemilattice $\mathscr{L}$ of $\mathscr{D}$ yields a closure operator $\mathrm{cl} = \mathrm{cl}[\mathscr{L}]$ on $\mathscr{D}$ defined by

$$\mathrm{cl}(A) := \bigwedge \{X \in \mathscr{L} \colon X \geqslant A\}.$$

These processes are mutually inverse in the sense that $\mathscr{L}[\mathrm{cl}[\mathscr{L}]] = \mathscr{L}$ and $\mathrm{cl}[\mathscr{L}[\mathrm{cl}]] = \mathrm{cl}$. All of that is well known.

Apparently novel is the following correspondence between $\wedge$-subsemilattices $\mathscr{L} \subseteq \mathscr{D}$ and "weakly submodular" rank functions on $\mathscr{D}$. Some definitions beforehand. A *map* is a monotone function $f \colon \mathscr{D} \to \mathbb{N}$ which satisfies $f(0) = 0$. The map f is:

$(R3)^+$ *modular*, if $f(A \vee B) + f(A \wedge B) = f(A) + f(B)$;
$(R3)$ *submodular*, if $f(A \vee B) - f(B) \leqslant f(A) - f(A \wedge B)$;
$(R3)^-$ *weakly submodular*, if $f(A) = f(A \wedge B)$ implies $f(A \vee B) = f(B)$;
$(R3)^{--}$ *locally submodular*, if it follows from $A \succ A \wedge B \prec B$ and $f(A) = f(A \wedge B) = f(B)$ that $f(A \vee B) = f(A \wedge B)$.

The stated identities and inequalities are supposed to hold for all $A, B \in \mathscr{D}$. Obviously

$$\text{locally submodular} \;\Rightarrow\; \text{weakly submodular} \;\Rightarrow\; \text{submodular} \;\Rightarrow\; \text{modular}.$$

A *rank function* is a map which satisfies $r(A) \leqslant |A|$ for all $A \in \mathscr{D}$. Here $|A|$ is the *natural rank* (or *height*) of A, i.e. the length n of a longest chain $0 \prec A_1 \prec A_2 \prec \cdots \prec A_n = A$. All *unit increase* maps r, i.e. $A \prec B \Rightarrow r(B) \leqslant r(A) + 1$, are easily seen to be rank functions. Here $\prec$ denotes the covering relation. We do not assume that $\mathscr{D}$ is graded.

Given a map $f \colon \mathscr{D} \to \mathbb{N}$, associate with it the set $\mathscr{L}[f]$ of all *f-maximal* elements $X \in \mathscr{D}$, thus

$$\mathscr{L}[f] := \{X \in \mathscr{D} \colon (\forall A \in \mathscr{D})\ (A > X \Rightarrow f(A) > f(X))\}.$$

Given a $\wedge$-subsemilattice $\mathscr{L}$ of $\mathscr{D}$ or, equivalently, a closure operator cl: $\mathscr{D} \to \mathscr{D}$, define a rank function $r = r[\mathscr{L}] = r[\mathrm{cl}]$ from $\mathscr{D}$ to $\mathbb{N}$ by

$$r(A) := \|\mathrm{cl}(A)\|,$$

where $\| \ \|$ gives the height *within* $\mathscr{L}$. Again, we do not assume that $\mathscr{D}$ or $\mathscr{L}$ are graded. Given a $\wedge$-subsemilattice $\mathscr{L}$ of $\mathscr{D}$ one checks that $\mathscr{L}[r[\mathscr{L}]] = \mathscr{L}$. More interesting is the question: given a map $f \colon \mathscr{D} \to \mathbb{N}$, when is $\mathscr{L}[f]$ a $\wedge$-subsemilattice?

Theorem 1 (*Wild [18, Theorem 1]*). *Let $\mathscr{D}$ be any lattice.*

(a) *If the map $f \colon \mathscr{D} \to \mathbb{N}$ is weakly submodular, then $\mathscr{L}[f]$ is a $\wedge$-subsemilattice of $\mathscr{D}$.*
(b) *If $\mathscr{L}$ is a $\wedge$-subsemilattice of $\mathscr{D}$, then $r[\mathscr{L}]$ is a weakly submodular rank function on $\mathscr{D}$.*