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Nordhaus–Gaddum results for restrained domination and total restrained domination in graphs

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set if every vertex is adjacent to a vertex in S and every vertex of $V - S$ is adjacent to a vertex in $V - S$. A set $S \subseteq V$ is a restrained dominating set if every vertex in $V - S$ is adjacent to a vertex in S and to a vertex in $V - S$. The total restrained domination number of G (restrained domination number of G, respectively), denoted by $\gamma_{tr}(G)$ ($\gamma_r(G)$, respectively), is the smallest cardinality of a total restrained dominating set (restrained dominating set, respectively) of G. We bound the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds. It is known (see [G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, L.R. Markus, Restrained domination in graphs, Discrete Math. 203 (1999) 61–69.]) that if G is a graph of order $n \geqslant 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then $4 \leqslant \gamma_r(G) + \gamma_r(\overline{G}) \leqslant n + 2$. We also provide characterizations of the extremal graphs G of order n achieving these bounds.

Keywords: Restrained; Total; Domination; Nordhaus–Gaddum

1. Introduction

In this paper, we follow the notation of [1]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a *dominating set*, denoted DS, of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [6,7].

In this paper, we continue the study of two variations of the domination theme, namely that of restrained domination [4,3,5,8] and total restrained domination [2,11].

A set $S \subseteq V$ is a *total restrained dominating set*, denoted TRDS, if every vertex is adjacent to a vertex in S and every vertex in $V - S$ is also adjacent to a vertex in $V - S$. Every graph without isolated vertices has a total restrained dominating set, since $S = V$ is such a set. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G.

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A set $S \subseteq V$ is a *restrained dominating set*, denoted RDS, if every vertex in $V - S$ is adjacent to a vertex in S and a vertex in $V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of G. If u, v are vertices of G, then the distance between u and v will be denoted by $d(u, v)$.

Nordhaus and Gaddum present best possible bounds on the sum of the chromatic number of a graph and its complement in [10]. The corresponding result for the domination number is presented by Jaeger and Payan in [9]: If G is a graph of order $n \geqslant 2$, then $\gamma(G) + \gamma(\overline{G}) \leqslant n + 1$. A best possible bound on the sum of the restrained domination numbers of a graph and its complement is obtained in [3]:

Theorem 1. *If G is a graph of order $n \geqslant 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then* $4 \leqslant \gamma_r(G) + \gamma_r(\overline{G}) \leqslant n + 2$.

A best possible bound on the sum of the total restrained domination numbers of a graph and its complement is obtained in [2]:

Theorem 2. *If G is a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices or has diameter two, then* $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 4$.

Let K be the graph obtained from K_3 by matching the vertices of $\overline{K}_2$ to distinct vertices of K_3. Note that K is self-complementary, K nor $\overline{K}$ contains isolated vertices or has diameter two, while $\gamma_{tr}(K) + \gamma_{tr}(\overline{K}) = 2 \times 5 = 10 > n(K) + 4$. Thus, Theorem 2 is incorrect.

We will show, in Section 2, that if G is a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K, then $4 \leqslant \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 4$. Moreover, we will characterize the graphs G of order n for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$ and also characterize those graphs G for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. In Section 3, we characterize the graphs G of order n for which $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ as well as those graphs G for which $\gamma_r(G) + \gamma_r(\overline{G}) = 4$.

2. Total restrained domination

In this section, we provide bounds on the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds.

Let $n \geqslant 5$ be an integer and suppose $\{x, y, u, v\}$ and X are disjoint sets of vertices such that $|X| = n - 4$. Let $\mathscr{L}$ be the family of graphs G of order n where $V(G) = \{x, y, u, v\} \cup X$ and with the following properties:

(P1) x and y are non-adjacent, while u and v are adjacent;
(P2) each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{u, v\}$;
(P3) each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{x, y\}$;
(P4) each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{x, y\} \cup X$;
(P5) each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{u, v\} \cup X$.

Theorem 3. *If G is a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices, then* $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$ *if and only if* $G \in \mathscr{L}$.

Proof. Suppose G is a graph such that neither G nor $\overline{G}$ contains isolated vertices, and suppose $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. Then $\gamma_{tr}(G) = \gamma_{tr}(\overline{G}) = 2$. Let $S = \{u, v\}$ ($S' = \{x, y\}$, respectively) be a TRDS of G ($\overline{G}$, respectively). Then x is non-adjacent to y, while u is adjacent to v, and Property (P1) holds. Clearly, $S \neq S'$. Suppose $u = x$ with $v \neq y$. Since $\{u, v\}$ is a DS of G and y is non-adjacent to $x = u$, the vertex y must be adjacent to v. But then v is not dominated by S' in $\overline{G}$, which is a contradiction. Thus, $S \cap S' = \emptyset$. Let $X = V(G) - \{x, y, u, v\}$. Then $|X| = n - 4$, and since S (S', respectively) is a TRDS of G ($\overline{G}$, respectively), Properties (P2)–(P5) hold for G. Thus, $G \in \mathscr{L}$. The converse clearly holds as $\{u, v\}$ ($\{x, y\}$, respectively) is a TRDS of G ($\overline{G}$, respectively). $\square$