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# A few more Kirkman squares and doubly near resolvable BIBDs with block size 3

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**Abstract**

A Kirkman square with index $\lambda$, latinicity $\mu$, block size $k$, and $v$ points, $\mathrm{KS}_k(v; \mu, \lambda)$, is a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) defined on a $v$-set $V$ such that (1) every point of $V$ is contained in precisely $\mu$ cells of each row and column, (2) each cell of the array is either empty or contains a $k$-subset of $V$, and (3) the collection of blocks obtained from the non-empty cells of the array is a $(v, k, \lambda)$-BIBD. In a series of papers, Lamken established the existence of the following designs: $\mathrm{KS}_3(v; 1, 2)$ with at most six possible exceptions [E.R. Lamken, The existence of doubly resolvable $(v, 3, 2)$-BIBDs, J. Combin. Theory Ser. A 72 (1995) 50–76], $\mathrm{KS}_3(v; 2, 4)$ with two possible exceptions [E.R. Lamken, The existence of $\mathrm{KS}_3(v; 2, 4)s$, Discrete Math. 186 (1998) 195–216], and doubly near resolvable $(v, 3, 2)$-BIBDs with at most eight possible exceptions [E.R. Lamken, The existence of doubly near resolvable $(v, 3, 2)$-BIBDs, J. Combin. Designs 2 (1994) 427–440]. In this paper, we construct designs for all of the open cases and complete the spectrum for these three types of designs. In addition, Colbourn, Lamken, Ling, and Mills established the spectrum of $\mathrm{KS}_3(v; 1, 1)$ in 2002 with 23 possible exceptions. We construct designs for 11 of the 23 open cases.


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## 1. Introduction

A balanced incomplete block design (BIBD) $D$ is a collection $B$ of subsets (blocks) taken from a finite set $V$ of $v$ elements with the properties:

(1) Every pair of distinct elements of $V$ is contained in precisely $\lambda$ blocks of $B$.
(2) Every block contains exactly $k$ elements.

We denote such a design as a $(v, k, \lambda)$-BIBD. The necessary conditions for the existence of a $(v, k, \lambda)$-BIBD are

$$\lambda(v-1) \equiv 0 \pmod{k-1} \quad \text{and} \quad \lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \tag{*}$$

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| | | | | |
|---|---|---|---|---|
| 0,1,2 | 0,3,4 | ∞,3,1 | | ∞,2,4 |
| ∞,3,0 | 1,2,3 | 1,4,0 | ∞,4,2 | |
| | ∞,4,1 | 2,3,4 | 2,0,1 | ∞,0,3 |
| ∞,1,4 | | ∞,0,2 | 3,4,0 | 3,1,2 |
| 4,2,3 | ∞,2,0 | | ∞,1,3 | 4,0,1 |

Fig. 1. A $KS_3(6; 2, 4)$, [11].

A $(v, k, \lambda)$-BIBD $D$ is said to be $\mu$-resolvable if the blocks of $D$ can be partitioned into classes $R_1, R_2, \ldots, R_t$ (resolution classes) where $t = \lambda(v-1)/\mu(k-1)$ such that each element of $D$ is contained in precisely $\mu$ blocks of each class. The classes $R_1, R_2, \ldots, R_t$ form a resolution of $D$. If $\mu=1$, the design is said to be resolvable and is usually denoted by $(v, k, \lambda)$-RBIBD. The necessary conditions for the existence of a $(v, k, \lambda)$-RBIBD are $(*)$ and $v \equiv 0 \pmod k$.

A $(v, k, \lambda)$-BIBD is said to be near resolvable if the blocks of $D$ can be partitioned into classes (resolution classes) $R_1, R_2, \ldots, R_v$ such that for each element $x$ of $D$ there is precisely one class which does not contain $x$ in any of its blocks and each class contains precisely $v-1$ distinct elements of the design. The classes $R_1, R_2, \ldots, R_v$ form a resolution of $D$ and $D$ is denoted by NR$(v, k, \lambda)$-BIBD. Two necessary conditions for the existence of an NR$(v, k, \lambda)$-BIBD are $v \equiv 1 \pmod k$ and $\lambda = k-1$.

Let $R$ and $R'$ be two resolutions of the blocks of a $(v, k, \lambda)$-BIBD $D$. $R$ and $R'$ are said to be orthogonal if $|R_i \cap R'_j| \leqslant 1$ for all $R_i \in R$ and $R'_j \in R'$. (It should be noted that the blocks of the design are considered as labeled so that if a subset of the elements occurs as a block more than once, the blocks are treated as distinct.) If $D$ is a $(v, k, \lambda)$-RBIBD with a pair of orthogonal resolutions, it is called doubly resolvable and is denoted by DR$(v, k, \lambda)$-BIBD. If $D$ is an NR$(v, k, \lambda)$-BIBD with a pair of orthogonal near resolutions, it is called doubly near resolvable and is denoted by DNR$(v, k, \lambda)$-BIBD.

The existence of a $\mu$-resolvable $(v, k, \lambda)$-BIBD with a pair of orthogonal $\mu$-resolutions is equivalent to the existence of a Kirkman square, $KS_k(v; \mu, \lambda)$ [8,14,15]. In particular, the existence of a DR$(v, k, \lambda)$-BIBD is equivalent to the existence of a $KS_k(v; 1, \lambda)$.

A Kirkman square with index $\lambda$, latinicity $\mu$, block size $k$, and $v$ points, $KS_k(v; \mu, \lambda)$, is a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) defined on a $v$-set $V$ such that

(1) every point of $V$ is contained in precisely $\mu$ cells of each row and column,
(2) each cell of the array is either empty or contains a $k$-subset of $V$, and
(3) the collection of blocks obtained from the non-empty cells of the array is a $(v, k, \lambda)$-BIBD.

We can use a pair of orthogonal $\mu$-resolutions of a $(v, k, \lambda)$-BIBD to construct a $KS_k(v; \mu, \lambda)$. We index the rows and columns of a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) with a pair of orthogonal resolutions, $R$ and $R'$. In the cell labeled $(R_i, R'_j)$, we place the block from $R_i \cap R'_j$ for all $R_i \in R$ and $R'_j \in R'$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. It is easy to verify that this array is a $KS_k(v; \mu, \lambda)$. Similarly, it is easy to see that a $KS_k(v; \mu, \lambda)$ displays a pair of orthogonal $\mu$-resolutions of a $(v, k, \lambda)$-BIBD. To illustrate these definitions, a $KS_3(6; 2, 4)$ is displayed in Fig. 1.

Similarly, a pair of orthogonal resolutions of a DNR$(v, k, \lambda)$-BIBD can be used to construct a $v \times v$ array. (For convenience, we often refer to this array as a DNR$(v, k, \lambda)$-BIBD.) We index the rows and columns of the array with the pair of orthogonal resolutions $R$ and $R'$. In the cell labeled $(R_i, R'_j)$, we place the block from $R_i \cap R'_j$ for all $R_i \in R$ and $R'_j \in R'$. If $R_i \cap R'_j = \emptyset$, the cell is left empty. The rows of the array will contain the resolution classes of the resolution $R$ and the columns will contain the resolution classes of the orthogonal resolution $R'$. If the DNR$(v, k, \lambda)$-BIBD has the additional property that under an appropriate ordering of the resolution classes $R$ and $R'$, $R_i \cup R'_i$ contains precisely $v-1$ distinct elements of the design and $R_i \cap R'_i = \emptyset$ for all $i$, then the array is called a $(1, \lambda; k, v, 1)$-frame. The diagonal of a $(1, \lambda; k, v, 1)$-frame is empty and a unique element of the design can be associated with each cell $(i, i)$. The DNR(10, 3, 2)-BIBD in Fig. 2 is a $(1, 2; 3, 10, 1)$-frame. The element associated with cell $(i, i)$ is $i$ for $i = 0, 1, \ldots, 9$. We note that it is not always possible to permute the rows and columns of a DNR$(v, k, \lambda)$-BIBD to form a $(1, \lambda; k, v, 1)$-frame; we refer to [13] for examples off DNR$(v, k, \lambda)$-BIBDs which are not $(1, \lambda; k, v, 1)$-frames. (The distinction between DNR$(v, k, \lambda)$-BIBDs and $(1, \lambda; k, v, 1)$-frames is important in recursive constructions.)