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Graphical complexity of products of permutation groups☆

Mariusz Grech, Artur Jeż, Andrzej Kisielewicz

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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Abstract

In this paper we study representations of permutation groups as automorphism groups of colored graphs and supergraphs. In particular, we consider how such representations for various products of permutation groups can be obtained from representations of factors and how the degree of complexity increases in such constructions.

Keywords: Colored graph; Automorphism group; Permutation group; Supergraph; Direct product; Wreath product

We investigate the representability of permutation groups by means of symmetries of graphical structures. In general, by graphical complexity of a permutation group P we mean the degree of complexity of a graphical structure whose symmetry group (automorphism group) is P. In this paper we consider graphs with colored edges and supergraphs.

It is well known that while every abstract group is isomorphic to the automorphism group of a graph, not every permutation group can be represented directly as the automorphism group of a concrete graph. Also, colored graphs do not suffice to this aim. In [7], it has been shown that fairly simple graphical structures whose automorphism groups include all permutation groups are supergraphs.

Supergraphs can be viewed as ordinary graphs which apart from edges (called superedges of rank 1) may have additional superedges of rank $k > 1$ joining superedges of lower rank (see definition below).

The *graphical complexity* $\mathrm{gc}(P)$ of a permutation group P is the least rank of a supergraph whose automorphism group is P. The class of all permutation groups P with $\mathrm{gc}(P) = k$ will be denoted $\mathrm{SGR}(k)$. By $\mathrm{GR}(k)$ we denote the class of the automorphism groups of k-(edge) colored graphs, and by $\mathrm{DGR}(k)$ the class of the automorphism groups of k-(edge) colored directed graphs. Investigation of these classes is intended as a new approach to the study of concrete permutation groups initiated in [11] and another attempt to understand better the structure of such groups. We would like to find a reason of high graphical complexity. In particular, a natural question which remains open is whether there are permutation groups in $\mathrm{GR}(k)\backslash\mathrm{GR}(k-1)$ for arbitrary large k. In this paper we try to do a step in this very direction investigating whether the complexity classes above are closed under various natural products of permutation groups.

The paper is organized as follows. More precise definitions and further comments are given in Section 1. In Section 2, we consider constructions of direct sum and direct product of permutation groups. Section 3 deals with the wreath product in its imprimitive actions, and the final Section 4 deals with the wreath product in the product action. Generally, we prove that the classes $\mathrm{SGR}(k)$ are closed under each of these constructions, which shows that the considered

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E-mail addresses: mgrec@math.uni.wroc.pl (M. Grech), kisiel@math.uni.wroc.pl (A. Kisielewicz).

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constructions do not increase graphical complexity of permutation groups. Yet, the classes $GR(k)$ fail to be closed in general, and in view of our results, the possibility that some of these construction may be used to obtain members of $GR(k)\backslash GR(k-1)$ for larger k, remains open.

1. Preliminaries

The terminology used in this paper is standard. The reader is referred to [4,6,7,9,10] for earlier results of our research group and to [1,2,12] for more general background. In this study permutation groups are finite and are considered up to *permutation isomorphism*: two permutation groups P and Q on the sets V and W, respectively, are treated as identical if there is a bijection $\alpha: V \to W$ such that after the identification of corresponding elements both the groups as the collections of permutations coincide.

A *k-colored graph* is a pair $\mathscr{G}=(V, E)$, where V is the set of vertices, and E a function from the set $P_2(V)$ of unordered pairs into the set of colors $\{0, 1, \ldots, k-1\}$, called the *edge-color function*. For a *k-colored digraph* the domain of the edge-color function E is the set $V \times V$ of ordered pairs. Thus, a colored digraph, in contrast with a colored graph, has loops by definition, which may be viewed as coloring vertices. Automorphisms are those permutations σ that preserve the edge function; in symbols: $E(v, w) = E(v\sigma, w\sigma)$, for all $v, w \in V$. (We use multiplication from the right to denote actions.)

A *supergraph* is a pair $\mathscr{H} = (V, E)$, where V is a set of vertices, and $E \subseteq E_k$ for some $k \geqslant 1$, with $E_1 = P_2(V)$, and $E_{k+1} = P_2(V \cup E_k)$ for all k; we require that E satisfies the following heridetarity condition: if $\{s, t\} \in E$, and s is not a vertex, then $s \in E$. The elements of $E_k \backslash E_{k-1}$ are called the *superedges of rank k*; the maximum of ranks of superedges is the rank of the supergraph. An automorphism of a supergraph $\mathscr{H} = (V, E)$ is the permutation σ of the vertices such that $\{s, t\} \in E$ if and only if $\{s\sigma, t\sigma\} \in E$, where the action of σ is extended on the superedges in the natural way.

The group of automorphisms of a graph (digraph, supergraph) G is denoted $\mathrm{Aut}(\mathscr{G})$. If a permutation group P is in $GR(k)$ (or $DGR(k)$, $SGR(k)$) we say that P is *k-representable* by a colored graph (colored digraph, supergraph). The graph (digraph, supergraph) in question is said also to represent P in such a case. We notice that in Wielandt's terminology [11] the permutation groups in the union DGR of all $DGR(k)$ are called 2-*closed*, and those in the union GR of all $GR(k)$ are called 2^*-*closed*. By the result of [7], the union of all $SGR(k)$ contains all permutation groups.

For example, no double transitive group is in DGR, with exception of the symmetry groups S_n which are representable by simple graphs. In fact, the symmetry groups form the class $GR(1) = DGR(1)$. Dihedral groups and one-generated groups belong to $DGR(2)$, but not all of them are in $GR(2)$ (see [4,7]). Observe that $GR(k) \subseteq DGR(k) \subseteq SGR(k)$ for all $k \geqslant 2$. In [7] it is proved that $GR(k) \subseteq SGR(2)$ and $DGR(k) \subseteq SGR(3)$ for all k.

We also define $GR^*(k) = GR(k)\backslash GR(k-1)$ for $k \geqslant 2$, and $GR^*(1) = GR(1)$. Groups in $GR^*(k)$ are called *strictly k-representable* by a colored graph. We introduce analogous notions for colored digraphs and supergraphs.

The main problem in this area is whether the classes $GR(k)$ and $DGR(k)$ form real hierarchies of 2-closed permutation groups, that is, whether sets $GR^*(k)$ and $DGR^*(k)$ are nonempty for all k. So far we know only that this is true for $k \leqslant 6$ for graphs, and $k \leqslant 5$ for digraphs (cf. [5]). For supergraphs, it has been proved that $SGR(k)$ form a strictly ascending chain (but see [7] for another form of the problem above and other open problems).

The study of the automorphism groups of supergraphs is facilitated by the fact that superedges come in a variety of types that have to be preserved by automorphisms. We say that a superedge $e = \{s, t\}$ in a supergraph S is *of type* (m, n) if the superedges s and t are of ranks m and n, respectively. The *degree of superedge e* is the number of superedges outcoming from e (i.e. the number of superedges of the form $\{e, r\} \in S$, where r is any superedge). The outcoming edges may be further partitioned with respect to types leading to relative degrees, and more types.

2. Direct products

Given two permutation groups P, Q acting on sets V and W, respectively, by the *direct sum* $P \oplus Q$ of P and Q we mean the permutation group on the disjoint union of V and W whose elements are ordered pairs (p, q) with $p \in P$, $q \in Q$, and the action is given by $v(p, q) = vp$ for $v \in V$ and $w(p, q) = wq$ for $w \in W$. (This construction is often called the “direct product”, and so was called in [4,9]. Yet, when we consider various actions of the direct product of abstract groups, and generally, from the point of view of concrete permutation groups, the term “direct sum”, used e.g. in [8], seems more appropriate.)