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Discrete Mathematics 308 (2008) 1153–1164
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Radio number for trees

Daphne Der-Fen Liu[1]

Department of Mathematics, California State University, Los Angeles, Los Angeles, CA 90032, USA

Received 17 August 2005; received in revised form 18 March 2007; accepted 27 March 2007
Available online 6 April 2007

Abstract

Let G be a connected graph with diameter $\mathrm{diam}(G)$. The *radio number* for G, denoted by $\mathrm{rn}(G)$, is the smallest integer k such that there exists a function $f : V(G) \to \{0, 1, 2, \ldots, k\}$ with the following satisfied for all vertices u and v: $|f(u) - f(v)| \geqslant \mathrm{diam}(G) - d_G(u, v) + 1$, where $d_G(u, v)$ is the distance between u and v. We prove a lower bound for the radio number of trees, and characterize the trees achieving this bound. Moreover, we prove another lower bound for the radio number of spiders (trees with at most one vertex of degree more than two) and characterize the spiders achieving this bound. Our results generalize the radio number for paths obtained by Liu and Zhu.

Keywords: Channel assignment problem; Distance-two labeling; Multi-level distance labeling; Radio number

1. Introduction

Multi-level distance labeling (or radio labeling) can be regarded as an extension of distance-two labeling, and both of them are motivated by the channel assignment problem introduced by Hale [8]. Given a set of stations (or transmitters), a *valid channel assignment* is a function that assigns to each station with a channel (nonnegative integer) such that interference is avoided. The task is to find a valid channel assignment with the minimum span of the channels used. The degree (or level) of interference is related to the locations of the stations—the closer the two stations, the stronger the interference that might occur. In order to avoid interference, the separation between the channels assigned to a pair of near-by stations must be large enough; the amount of the required separation depends on the distance between the two stations.

A graph model for this problem is to represent each station by a vertex, and connect every pair of close stations by an edge. Let G be a connected graph. We denote the distance between two vertices u and v by $d_G(u, v)$, or $d(u, v)$ if G is clear in the context. Motivated by the channel assignment problem with two levels of interference, a *distance-two labeling* for G is a function $f : V \to \{0, 1, 2, 3, \ldots\}$ such that $|f(u) - f(v)| \geqslant 2$ if $d(u, v) = 1$, and $f(u) \neq f(v)$ if $d(u, v) = 2$. The *span* of f is defined as $\max_{u,v\in V}\{f(u) - f(v)\}$. The λ-number for a graph G, denoted by $\lambda(G)$, is the minimum span of a distance-two labeling for G. Distance-two labeling has been studied intensively in the past decade (cf. [1,2,5–7,9–12,16]).

E-mail address: dliu@calstatela.edu.
[1] Supported in part by the National Science Foundation under Grant DMS0302456.

doi:10.1016/j.disc.2007.03.066

Motivated by the channel assignment problem with $\mathrm{diam}(G)$ levels of interference, a *multi-level distance labeling* (or *radio labeling*) is a function $f : V(G) \to \{0, 1, 2, 3, \ldots\}$ so that the following is satisfied for $u, v \in V(G)$:

$$|f(u) - f(v)| \geqslant \mathrm{diam}(G) - d(u, v) + 1,$$

where $\mathrm{diam}(G)$ is the *diameter* of G (the maximum distance over all pairs of vertices). The *radio number* (as suggested by the FM radio channel assignment [4]) for a graph G, denoted by $\mathrm{rn}(G)$, is the minimum span of a radio labeling for G. Note that when $\mathrm{diam}(G)=2$, distance-two labeling coincides with radio labeling, and in this case, $\lambda(G)=\mathrm{rn}(G)$.

Finding the radio number for a graph is an interesting yet challenging task. So far, the value is known only to very limited families of graphs. For paths and cycles, it was studied by Chartrand et al. [4,3] and Zhang [17], while the exact value remained open until lately solved by Liu and Zhu [13]. The radio number for the *square* (adding edges between vertices of distance two apart) of paths was completely determined by Liu and Xie [14] who also studied the problem for the square of cycles [15].

The aim of this article is to extend the study to trees. In Section 2, we prove a general lower bound for the radio number of trees and characterize the trees achieving this bound. Then we focus on the study of a special family of trees called *spiders* which are trees with at most one vertex of degree more than two. Besides the lower bound obtained by applying the result of trees to spiders, in Section 3, we present another lower bound for spiders and characterize the spiders achieving those bounds.

2. A lower bound for trees

As we are seeking for the minimum span of a radio labeling for a graph G, without loss of generality, we always assume that the label 0 is used by any radio labeling f. So the span of f is the maximum label used. A radio labeling for G with span equal to $\mathrm{rn}(G)$ is called an *optimal radio labeling*.

Let T be a tree rooted at a vertex w. For any two vertices u and v, if u is on the (w, v)-path, then u is an *ancestor* of v, and v is a *descendent* of u. The root w is an ancestor of every vertex, and every vertex is its own ancestor and descendent. Fix any w as the root, define the *level function* on $V(T)$ by

$$L_w(u) = d(w, u) \quad \text{for any } u \in V(T).$$

For any $u, v \in V(T)$, define

$$\phi_w(u, v) = \max\{L_w(t) : t \text{ is a common ancestor of } u \text{ and } v\}.$$

Let w' be a neighbor of w. We call the subtree induced by w' together with all the descendents of w' a *branch*.

Observation 1. *Let T be a tree rooted at w. For any vertices u and v,*

(1) $\phi_w(u, v) = 0$ *if and only if u and v belong to different branches (unless one of them is w), and*
(2) $d(u, v) = L_w(u) + L_w(v) - 2\phi_w(u, v)$.

For any vertex w in a tree T, the *weight of T rooted at w* is defined by

$$w_T(w) = \sum_{u \in V(T)} L_w(u).$$

The *weight of T* is the smallest weight among all possible roots of T:

$$w(T) = \min\{w_T(w) : w \in V(T)\}.$$

A vertex w^* of a tree T is called a *weight center* of T if $w_T(w^*) = w(T)$.

If ww' is an edge of T and $T_w, T_{w'}$ are two components of $T - ww'$, then it follows easily from the definition that $\omega_T(w) = \omega_T(w') + |V(T_{w'})| - |V(T_w)|$. Therefore, the next two lemmas emerge.

Lemma 1. *Suppose w^* is a weight center of a tree T. Then each component of $T - w^*$ contains at most $|V(T)|/2$ vertices.*