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Matching properties in connected domination critical graphs ☆

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Abstract

A dominating set of vertices S of a graph G is connected if the subgraph $G[S]$ is connected. Let $\gamma_c(G)$ denote the size of any smallest connected dominating set in G. A graph G is k-γ-connected-critical if $\gamma_c(G) = k$, but if any edge $e \in E(\bar{G})$ is added to G, then $\gamma_c(G+e) \leqslant k-1$. This is a variation on the earlier concept of criticality of edge addition with respect to ordinary domination where a graph G was defined to be k-critical if the domination number of G is k, but if any edge is added to G, the domination number falls to $k-1$.

A graph G is factor-critical if $G-v$ has a perfect matching for every vertex $v \in V(G)$, bicritical if $G-u-v$ has a perfect matching for every pair of distinct vertices $u, v \in V(G)$ or, more generally, k-factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. In two previous papers [N. Ananchuen, M.D. Plummer, Matching properties in domination critical graphs, Discrete Math. 277 (2004) 1–13; N. Ananchuen, M.D. Plummer, 3-factor-criticality in domination critical graphs, Discrete Math. 2007, to appear [3].] on ordinary (i.e., not necessarily connected) domination, the first and third authors showed that under certain assumptions regarding connectivity and minimum degree, a critical graph G with (ordinary) domination number 3 will be factor-critical (if $|V(G)|$ is odd), bicritical (if $|V(G)|$ is even) or 3-factor-critical (again if $|V(G)|$ is odd). Analogous theorems for connected domination are presented here. Although domination and connected domination are similar in some ways, we will point out some interesting differences between our new results for the case of connected domination and the results in [N. Ananchuen, M.D. Plummer, Matching properties in domination critical graphs, Discrete Math. 277 (2004) 1–13; N. Ananchuen, M.D. Plummer, 3-factor-criticality in domination critical graphs, Discrete Math. 2007, to appear [3].].

Keywords: Connected domination; Critical edge; Matching; Factor-critical; Bicritical; 3-factor-critical; Claw-free

1. Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is a *dominating set* for G if every vertex of G either belongs to S or is adjacent to a vertex of S. If S dominates G, we write $S \succ G$. The minimum cardinality of a dominating set in a graph G is called the *domination number* of G and is denoted by $\gamma(G)$. Graph G is said to be *k-γ-critical* if $\gamma(G) = k$, but $\gamma(G+e) = k-1$ for each edge $e \in E(\bar{G})$.

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A dominating set $S \subseteq V(G)$ is a *connected* dominating set if the subgraph spanned by S is connected. If S is a connected dominating set for G we write $S \succ_c G$. The minimum cardinality of a connected dominating set in G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. (Note that since a graph must be connected to have a connected dominating set, henceforth in this paper, when referring to connected domination, we shall assume all graphs under consideration are connected.) A graph G is *k-γ-connected critical* if $\gamma_c(G)) = k$, but $\gamma_c(G + uv) \leqslant k - 1$, for any edge $uv \in E(\bar{G})$. Note that while the addition of an edge may reduce the ordinary domination number by at most one, edge addition may reduce the connected domination number by at most two. (See Theorem 1 of [5].) In this paper, we will be concerned only with the case $k = 3$ and will refer to a connected-critical graph with connected domination number 3 as a *3-c-critical* graph.

The origins of the concept of connected domination are a bit hazy, although in the first published paper on the subject, Sampathkumar and Waliker [10] attribute the terminology to Hedetniemi. For a summary of their results, as well as a number of other early results on connected domination, see [7,8]. The algorithmic aspects of both domination and connected domination were first discussed by Garey and Johnson in their book [6] where it is claimed that both domination and connected domination are NP-complete, even when the graph is planar and regular of degree 4. For an excellent and more recent discussion of the computational and extremal aspects of connected domination, see [4].

More recently, Chen et al., [5] began the study of connected domination critical graphs by obtaining some results most of which have previous analogs for ordinary domination critical graphs. We will state and use several of their results below. Also following their notation, we will adopt the following. If u, v and w are vertices of G and $\{u, v\} \succ_c G - w$, but neither u nor v dominates w, we write $[u, v] \longrightarrow_c w$.

Following the work of Sumner and Blitch [11] on 3-critical graphs, Chen et al. [5] proved the following very useful result.

Lemma 1.1. *Let G be a 3-c-critical graph and let S be an independent set of $n \geqslant 3$ vertices in $V(G)$.*

(i) *then the vertices of S can be ordered as $a_1, a_2, \ldots, a_n$ in such a way that there exists a path of distinct vertices $x_1, x_2, \ldots, x_{n-1}$ in $G - S$ so that $[a_i, x_i] \longrightarrow_c a_{i+1}$ for $i = 1, 2, \ldots, n - 1$, and*
(ii) $\operatorname{diam}(G) \leqslant 3$.

The following lemma, may be viewed as being related to toughness. Proof of part (i) may be found in [5]. Part (ii) was later proved by the first author [1].

Lemma 1.2. *Let G be a 3-c-critical graph. Then*:

(i) *if T is a cutset of vertices for G, it follows that $G - T$ has at most $|T| + 1$ components, and moreover*;
(ii) *if the cutset T has at least two vertices, $G - T$ has at most $|T|$ components.*

Throughout the rest of this paper, $c(G)$ (respectively, $c_o(G)$) will denote the number of components (respectively, *odd* components) of a graph G. Also if G is a graph and $H \subseteq V(G)$, then $G[H]$ will denote the subgraph induced by H.

A *perfect* (respectively, *near-perfect*) matching in a graph G is a matching which covers all (respectively, all but one) of the vertices of G.

Lemma 1.3. *Let G be a 3-c-critical graph. Then*:

(i) *if $|V(G)|$ is even, G contains a perfect matching, while*;
(ii) *if $|V(G)|$ is odd, G contains a near-perfect matching.*

Proof. Part (i) is proved in [5]. We prove only part (ii). Suppose G is a 3-c-critical graph with an odd number of vertices and suppose G does not contain a near-perfect matching. Consider the Gallai–Edmonds decomposition of G. (See [9].) That is, let $D(G)$ denote the set of all vertices $v \in V(G)$ such that some maximum matching of G does not cover v. Let $A(G)$ denote the set of all neighbors of vertices of $D(G)$ which are not themselves in $D(G)$ and finally, let $C(G) = V(G) - (D(G) \cup A(G))$. Since G contains no near-perfect matching, then by Tutte's Theorem and parity,