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Bandwidth of the strong product of two connected graphs

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Abstract

The bandwidth $B(G)$ of a graph G is the minimum of the quantity $\max\{|f(x) - f(y)| : xy \in E(G)\}$ taken over all proper numberings f of G. The strong product of two graphs G and H, written as $G(\mathrm{S_P})H$, is the graph with vertex set $V(G) \times V(H)$ and with (u_1, v_1) adjacent to (u_2, v_2) if one of the following holds: (a) u_1 and v_1 are adjacent to u_2 and v_2 in G and H, respectively, (b) u_1 is adjacent to u_2 in G and $v_1 = v_2$, or (c) $u_1 = u_2$ and v_1 is adjacent to v_2 in H. In this paper, we investigate the bandwidth of the strong product of two connected graphs. Let G be a connected graph. We denote the diameter of G by $D(G)$. Let d be a positive integer and let x, y be two vertices of G. Let $N_G^{(d)}(x)$ denote the set of vertices v so that the distance between x and v in G is at most d. We define $\delta_d(G)$ as the minimum value of $|N_G^{(d)}(x)|$ over all vertices x of G. Let $N_G^{(d)}(x, y)$ denote the set of vertices z such that the distance between x and z in G is at most $d - 1$ and z is adjacent to y. We denote the larger of $|N_G^{(d)}(x, y)|$ and $|N_G^{(d)}(y, x)|$ by $\eta_G^{(d)}(x, y)$. We define $\eta(G) = 1$ if G is complete and $\eta(G)$ as the minimum of $\eta_G^{(D(G))}(x, y)$ over all pair of vertices x, y of G otherwise. Let G and H be two connected graphs. Among other results, we prove that if $\delta_{D(H)}(G) \geqslant B(G)D(H) + 1$ and $B(H) = \lceil (|V(H)| + \eta(H) - 2)/D(H) \rceil$, then $B(G(\mathrm{S_P})H) = B(G)|V(H)| + B(H)$. Moreover, we show that this result determines the bandwidth of the strong product of some classes of graphs. Furthermore, we study the bandwidth of the strong product of power of paths with complete bipartite graphs.

Keywords: Graph; Bandwidth; Strong product; Diameter; Distance

1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For two vertices $x, y \in V(G)$, let $d_G(x, y)$ denote the distance between x and y in G, and let $D(G)$ denote the diameter of G. Let $N_G(x)$ denote the neighborhood of a vertex x of G, and let $\deg_G(x)$ denote the degree of x in G. We write $\delta(G)$ and $\kappa(G)$ for the minimum degree and the connectivity of a graph G, respectively. We denote the path, the cycle, and the complete graph on n vertices by P_n, C_n, and K_n, respectively. Let $K_{m,n}$ denote the complete bipartite graph. We denote the kth power of a graph G by G^k.

Let G be a graph on n vertices. A one-to-one mapping $f : V(G) \to \{1, 2, \ldots, n\}$ is called a *proper numbering* of G. The *bandwidth of a proper numbering* f of G, denoted by $B_f(G)$, is the maximum difference between $f(x)$ and $f(y)$

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when xy runs over all edges of G, namely,

$$B_f(G) = \max\{|f(x) - f(y)| : xy \in E(G)\}.$$

The *bandwidth* of G is defined to be the minimum of $B_f(G)$ over all proper numberings f of G, and is denoted as $B(G)$, i.e.,

$$B(G) = \min\{B_f(G) : f \text{ is a proper numbering of } G\}.$$

For instance, $B(P_n^k) = k\ (n \geqslant k+1)$, $B(C_n^k) = 2k\ (n \geqslant 2k+1)$, $B(K_n) = n-1$, and $B(K_{m,n}) = \lceil m/2 \rceil + n - 1$ where $m \geqslant n$ (see [1,2,7]). A proper numbering f of G is called a *bandwidth numbering* of G when $B_f(G) = B(G)$.

The bandwidth problem for graphs arises from sparse matrix computation, coding theory, and circuit layout of VLSI designs. Papadimitriou [8] proved that the problem of determining the bandwidth of a graph is NP-complete, and Garey et al. [3] showed that it remains NP-complete even if graphs are restricted to trees with maximum degree three. Many studies have been done towards finding the bandwidth of specific classes of graphs (see [1,2,7]). In this paper, we investigate the bandwidth of the strong product of two connected graphs.

The *strong product* of two graphs G and H, written as $G(\mathrm{S_P})H$, is the graph whose vertex set is $V(G) \times V(H)$ with two vertices (u_1, v_1) and (u_2, v_2) adjacent if and only if ($u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$) or ($u_1u_2 \in E(G)$ and $v_1 = v_2$) or ($u_1 = u_2$ and $v_1v_2 \in E(H)$). There are some results on the bandwidth of the strong product of certain graphs.

Proposition 1. *Let m and n be positive integers.*

(i) $B(P_m(\mathrm{S_P})P_n) = n + 1$ *for* $m \geqslant n \geqslant 2$ [4,6].
(ii) $B(P_m(\mathrm{S_P})K_n) = 2n - 1$ *for* $m \geqslant 2$ *and* $n \geqslant 3$ [4].
(iii) $B(C_m(\mathrm{S_P})K_n) = 3n - 1$ *for* $m \geqslant 3$ *and* $n \geqslant 3$ [4].
(iv) $B(P_m(\mathrm{S_P})C_n) = \begin{cases} n+2 & \text{if } m \geqslant \lfloor n/2 \rfloor + 1 \\ 2m+1 & \text{otherwise} \end{cases}$ *for* $m \geqslant 2$ *and* $n \geqslant 3$[6].
(v) $B(C_m(\mathrm{S_P})C_n) = 2n + 2$ *for* $m \geqslant n \geqslant 3$ [6].

The following upper bound for the bandwidth of the strong product of two graphs is known.

Proposition 2 (*Hendrich and Stiebitz [4]*). *For any two graphs G and H,*

$$B(G(\mathrm{S_P})H) \leqslant \min\{B(G)|V(H)| + B(H),\ B(H)|V(G)| + B(G)\}.$$

We study the bandwidth of the strong product of two connected graphs which satisfy a few conditions. Let G be a connected graph on n vertices. Let d be a positive integer and let x, y be two vertices of G. We write $N_G^{(d)}(x)$ for the set of vertices v satisfying $d_G(x, v) \leqslant d$. Note that $N_G^{(1)}(x) = N_G(x) \cup \{x\}$. We define $\delta_d(G)$ as the minimum value of $|N_G^{(d)}(x)|$ taken over all $x \in V(G)$, i.e.,

$$\delta_d(G) = \min\{|N_G^{(d)}(x)| : x \in V(G)\}.$$

We remark that $\delta_1(G) = \delta(G) + 1$. Let $N_G^{(d)}(x, y)$ denote the set of vertices z so that $d_G(x, z) \leqslant d - 1$ and $zy \in E(G)$. We denote the larger of $|N_G^{(d)}(x, y)|$ and $|N_G^{(d)}(y, x)|$ by $\eta_G^{(d)}(x, y)$. We remark that $\eta_G^{(1)}(x, x) = 0$ and $\eta_G^{(d)}(x, x) = \deg_G(x)$ if $d \geqslant 2$. We define $\eta(G)$ as follows:

$$\eta(G) = \begin{cases} 1 & \text{if } G = K_n, \\ \min\{\eta_G^{(D(G))}(x, y) : x, y \in V(G) \text{ (not necessarily distinct)}\} & \text{otherwise.} \end{cases}$$

Note that $\delta(G) \geqslant \eta(G) \geqslant 1$ for any connected graph G. We get the following theorem, which gives a lower bound for the bandwidth of the strong product of two connected graphs. To state our results we use $\delta_d(G)$ and $\eta(G)$.