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Trees with depression three☆

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Abstract

An edge ordering of a graph $G=(V,E)$ is an injection $f:E\to\mathbb{N}$. A (simple) path for which f increases along its edge sequence is an f-ascent, and a maximal f-ascent if it is not contained in a longer f-ascent. The depression of G is the least integer k such that every edge ordering of G has a maximal ascent of length at most k. We characterise trees with depression three.

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1. Introduction

For concepts not defined here we refer the reader to [4]. The *neighbourhood* $N(v)$ of a vertex v of a simple graph $G=(V,E)$ is defined by $N(v)=\{u\in V : uv\in E\}$. An *edge ordering* of G is an injection $f:E\to\mathbb{N}$. Denote the set of all edge orderings of G by $\mathscr{F}(G)$. For any $f\in\mathscr{F}(G)$ a path a,b,c,d of length three such that $f(bc)=\min\{f(ab), f(bc), f(cd)\}$ or $f(bc)=\max\{f(ab), f(bc), f(cd)\}$ is called an *f-exchange*. A path λ in G for which $f\in\mathscr{F}(G)$ increases along its edge sequence is called an *f-ascent* (or simply *ascent* if the ordering is clear), and if λ has length k, it is also called a (k,f)-*ascent*. Thus an f-ascent contains no f-exchanges. If the path λ with vertex sequence $v_0, v_1, \ldots, v_k$ forms an f-ascent, we denote this fact by writing λ as $v_0v_1\ldots v_k$. An f-ascent is *maximal* if it is not contained in a longer f-ascent. Let $h(f)$ denote the length of a shortest maximal f-ascent and define the *depression* $\varepsilon(G)$ of G by

$$\varepsilon(G)=\max_{f\in\mathscr{F}(G)}\{h(f)\},$$

that is, $\varepsilon(G)$ is the smallest integer k such that every edge ordering of G has a maximal ascent of length at most k. To show that $\varepsilon(G)=k$, we must therefore show that:

(a) each edge ordering of G has a maximal ascent of length at most k—this shows that $\varepsilon(G)\leqslant k$,

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(b) there exists an edge ordering f of G with no maximal ascents of length less than k, i.e. for which each (l, f)-ascent, where $l < k$, can be extended to a (k, f)-ascent—this shows that $\varepsilon(G) \geqslant k$.

The study of the lengths of increasing paths in edge-ordered graphs was initiated by Chvátal and Komlós [5] who posed the problem of determining the *altitude* $\alpha(K_n)$, the greatest integer k such that K_n has a (k, f)-ascent for each edge ordering $f \in \mathscr{F}(K_n)$. They also considered the corresponding problem in the case where f-ascents are trails, not necessarily paths. However, with the exception of [8], subsequent work (see e.g. [1–3,9–11]) has focussed on the former problem. Note that these two concepts (f-ascents are paths versus f-ascents are trails) are equivalent for trees.

The depression of a graph was first defined in [6]. Clearly, $\varepsilon(G) = 1$ if and only if K_2 is a component of G. For any path u, v, w in a graph G, let $\tau(uvw)$ be the length of a longest path in G containing the subpath u, v, w. Define $\tau'(G) = \min\{\tau(uvw)\}$, where the minimum is taken over all paths of G of length two. As shown in [6], $\varepsilon(G) \leqslant \tau'(G)$ for all graphs G. It follows that if G has a vertex adjacent to two leaves, then $\varepsilon(G) = 2$. Graphs with depression two were characterised in [6].

Theorem 1 (*Cockayne et al. [6]*). *If G is connected, then $\varepsilon(G) = 2$ if and only if G has a vertex adjacent to two leaves or to two adjacent vertices of degree two.*

The purpose of this paper is to characterise trees with depression three.

2. A general result

Theorem 1 shows that there is no forbidden subgraph characterisation of graphs with depression two, because if any vertex of an arbitrary graph is joined to two new vertices, the resulting graph has depression two.

For two disjoint graphs G_1 and G_2 and vertices $v_i \in G_i$, if we identify v_1 and v_2 to form a new vertex v, we also say that we *attach G_2 to G_1* (or *G_1 to G_2*) *at v*. If G is the resulting graph, we say that G contains G_2 *as attachment* (*at v*). Thus, by Theorem 1, if v is any vertex of K_3 or the central vertex of P_3 and G is any graph that contains K_3 or P_3 as attachment at v, then $\varepsilon(G) = 2$.

Two interesting questions arise from this result. Firstly, if H is a graph with $\varepsilon(H) = k$ and $v \in V(H)$, what properties should H and v satisfy so that if we attach H to an arbitrary graph at v, the resulting graph has depression at most k? Secondly, for k fixed, can we find a minimal class $\mathscr{H}$ of graphs with depression k so that a graph G satisfies $\varepsilon(G) \leqslant k$ if and only if G contains some $H \in \mathscr{H}$ as attachment?

For example, $\varepsilon(H) = 3$ for $H \in \{P_4, C_4\}$, and if H is attached at one of its vertices of degree two to any graph, the resulting graph has depression at most three, as is shown below. On the other hand, $\varepsilon(C_5) = 3$ also (see [6]), but it is easy to show that if two copies of C_5 are attached to each other, then the resulting graph has depression four.

As it turns out, the first question is easy to answer and, in fact, to generalise. The generalisation shows that the second question is not quite the correct question to ask, as simply attaching a graph H to another graph is not the only operation to limit the depression of the resulting graph to at most that of H.

A *k-kernel*, or just *kernel* if k is unimportant, of a graph G with $\varepsilon(G) = k$ is a set $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal (l, f)-ascent for some $l \leqslant k$ that neither starts nor ends at a vertex in U. One part of the proof of Theorem 1 is based on the fact that the central vertex of P_3 is the (unique) 2-kernel of P_3, while any (single) vertex of K_3 forms a 2-kernel.

The next simple result is the key to solving the first question above and is therefore stated as a theorem.

Theorem 2. *Let U be a k-kernel of a graph H. Form a graph G by adding any set A of new vertices and arbitrary edges joining vertices in $U \cup A$. Then $\varepsilon(G) \leqslant \varepsilon(H)$.*

Proof. Consider any edge ordering f' of G and let f be the edge ordering of H induced by f'. Then, for some $l \leqslant k$, there is a maximal (l, f)-ascent λ in H that does not start or end at a vertex in U. Hence $\lambda = v_0 \ldots v_l$, where $v_0, v_l \in V(G) - (A \cup U)$. But then $N_G(v_0) = N_H(v_0)$ and $N_G(v_l) = N_H(v_l)$. Therefore λ is a maximal f'-ascent in G and so $h(f') \leqslant k$. Since f' is arbitrary, it follows that $\varepsilon(G) \leqslant k$. $\square$

If $\{v\}$ is a kernel of H, then the graph G described in Theorem 2 is obtained by attaching H to $\langle \{v\} \cup A \rangle$ at v. It is easy to ascertain that any vertex of C_4 and either vertex of P_4 of degree two is a 3-kernel. It follows that if P_4 or