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The polynomial degrees of Grassmann and Segre varieties over GF(2)

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Abstract

A recent proof that the Grassmannian $\mathscr{G}_{1,n,2}$ of lines of PG$(n, 2)$ has polynomial degree $\binom{n}{2} - 1$ is outlined, and is shown to yield a theorem about certain kinds of subgraphs of any (simple) graph $\Gamma = (\mathscr{V}, \mathscr{E})$ such that $|\mathscr{E}| < |\mathscr{V}|$. Somewhat similarly, the polynomial degree of the Segre variety $\mathscr{S}_{m,n,2}$, $m \leqslant n$, is shown to be $mn + m$, and in consequence a graph theory result is obtained about certain subgraphs of any graph Γ which is a subgraph of the complete bipartite graph $K_{m+1,n+1}$.

MSC: 51E20; 05C30; 05C90; 14G25

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1. The polynomial degree of a subset ψ of PG$(N, 2)$

In succeeding sections we will be interested in the polynomial degrees of the following varieties over the finite field GF(2):

(i) the Grassmann variety $\mathscr{G}_{1,n,2}$ of the lines of PG$(n, 2)$, considered as a subset of points of the finite projective space PG$(\binom{n+1}{2} - 1, 2) = \mathbb{P}(\wedge^2 V_{n+1,2})$;
(ii) the Segre variety $\mathscr{S}_{m,n,2}$, considered as a subset of points of the finite projective space PG$(mn + m + n, 2) = \mathbb{P}(V_{m+1,2} \otimes V_{n+1,2})$.

However, it will help to first consider material concerned with the polynomial degree of a general subset ψ of points of a general finite projective space PG$(N, 2) = \mathbb{P}(V)$, where $V = V_{N+1} = V(N + 1, 2)$.

For the most part the notation will be as in [11]. In particular $S = \mathrm{PG}^{(0)}(N, 2)$ denotes the set of points (0-flats) of PG$(N, 2) = \mathbb{P}(V)$, and we identify S with the nonzero vectors $V\backslash\{0\}$ of the vector space V. The set $F(V)$ of all functions $V \to$ GF(2) is a vector space over GF(2) of dimension $|V| = 2^{N+1}$, and its elements are the characteristic functions $\chi(\psi)$, also denoted χ_ψ, of the subsets $\psi \subseteq V$. In the case when ψ is a singleton set $\{a\}$, $a \in V$, we put

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$\chi_a := \chi_{\{a\}}$. In fact, rather than $F(V)$, *our main focus is on the vector subspace* $F(S)$, *of dimension* $|S| = 2^{N+1} - 1$ *over* GF(2), *consisting of all functions* $S \to$ GF(2).

Upon choosing a basis $\mathscr{B} = \{e_1, e_2, \ldots, e_{N+1}\}$ for V an element $x \in V$ may be viewed as an $(N+1)$-tuple $(x_1, x_2, \ldots, x_{N+1}) \in \mathrm{GF}(2)^{N+1}$. The basis $\mathscr{B}$ for V gives rise to an associated *monomial basis* $\mathscr{M}$ for $F(S)$, namely

$$\mathscr{M} = \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_{N+1} \quad \text{where } \Xi_r = \{x_{i_1} x_{i_2} \ldots x_{i_r}\}_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant N+1}. \tag{1.1}$$

If ψ^c denotes the complement *within the set* S of ψ then $\chi(\psi) + \chi(\psi^c) = I$, where I denotes that element of $F(S)$ such that $I(x) = 1$ for all $x \in S$. The characteristic functions χ_a, $a \in S$, have the coordinate expression:

$$\chi_a(x) = \chi_0(a + x) \quad \text{where } \chi_0(x) = \prod_{i=1}^{N+1} (1 + x_i), \tag{1.2}$$

and $I = \chi(S)$ has the coordinate expression

$$I(x) = 1 + \prod_{i=1}^{N+1} (1 + x_i) = \sum_i x_i + \sum_{i<j} x_i x_j + \cdots + x_1 x_2 \ldots x_{N+1}. \tag{1.3}$$

This last expression (1.3) may be viewed as the special case $r = N + 1$, $X^c = S$, of the following easily verified result:

if X is an $(N - r)$-flat in PG$(N, 2)$ which is the intersection of the r hyperplanes $f_1(x) = 0, \ldots, f_r(x) = 0$, then

$$\chi(X^c) = 1 + \prod_{i=1}^{r} (1 + f_i) = \sum_i f_i + \sum_{i<j} f_i f_j + \sum_{i<j<k} f_i f_j f_k + \cdots + f_1 f_2 \ldots f_r. \tag{1.4}$$

For $r > 0$, let $F_r = F_r(S)$ denote the subspace of $F(S)$ which consists of functions f expressible as a polynomial function $f(x_1, x_2, \ldots, x_{N+1})$ with $\deg f \leqslant r$ and $f(0) = 0$; we put $F_0 := \{0\}$. The subspaces F_r are thus nested:

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_N \subset F_{N+1} = F(S), \tag{1.5}$$

with F_r, $r \geqslant 1$, possessing the monomial basis $\mathscr{M}_r$ where

$$\mathscr{M}_r = \Xi_1 \cup \Xi_2 \cup \cdots \cup \Xi_r, \quad 1 \leqslant r \leqslant N + 1. \tag{1.6}$$

Observing that Ξ_{N+1} consists of the single monomial $m_{N+1} := x_1 x_2 \ldots x_{N+1}$, it follows from (1.2) that F_N consists of the characteristic functions of all even ($|\psi| \equiv 0 \bmod 2$) subsets ψ of S.

The subspace F_r of $F(S)$ has just been given an algebraic definition, but *there exists an equivalent geometric definition*, namely as that subspace of $F(S)$ which is generated by the characteristic functions $\chi(X^c)$ of the complements X^c of the $(N - r)$-flats X of PG$(N, 2)$. For if we define subspaces C_r, $0 \leqslant r < N$, of $F(S)$ by

$$C_r = \prec \chi(X^c) \succ_{X \in \mathrm{PG}^{(r)}(N,2)}, \tag{1.7}$$

then it can be shown, see [11, Theorem 1.5], cf. [1, Section 5.3], that

$$C_{N-r} = F_r, \quad r = 1, 2, \ldots, N. \tag{1.8}$$

Setting $Q_\psi := \chi(\psi^c)$, a subset ψ of S has equation $Q_\psi(x) = 0$. If $Q_\psi \in F_r \backslash F_{r-1}$ we will say that ψ has *polynomial degree* r, and we write $\deg Q_\psi = r$ for the degree of Q_ψ. (Here $\deg Q_\psi$ is the *reduced* degree of Q_ψ; if $\deg Q_\psi = r$ then of course, see (1.5), $Q_\psi \in F_s$ for each $s \geqslant r$.) Recall that the subspace $C_0 = F_N$ consists of the characteristic functions of all the even subsets of S. Consequently if ψ is an odd subset of S (and so ψ^c is an even subset) then ψ has polynomial degree $\leqslant N$. On the other hand, since $\chi(\psi) + \chi(\psi^c) = I$, and $\deg I = N + 1$, *an even subset always has polynomial degree* $N + 1$.

In general the determination of the polynomial degree of a subset $\psi \subset S$ is a formidable problem—and especially so if a direct algebraic approach is attempted, based for example upon (1.2). But quite often progress can be made by using a geometrical approach based upon the next theorem.