Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 307 (2007) 3006–3015
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

3-Factor-criticality in domination critical graphs

Nawarat Ananchuen[a,1], Michael D. Plummer[b,2]

[a]Department of Mathematics, Faculty of Science, Silpakorn University, Nakorn Pathom, Thailand
[b]Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

Received 3 December 2005; received in revised form 4 January 2007; accepted 11 March 2007
Available online 15 March 2007

Abstract

A graph G is said to be k-γ-critical if the size of any minimum dominating set of vertices is k, but if any edge is added to G the resulting graph can be dominated with $k-1$ vertices. The structure of k-γ-critical graphs remains far from completely understood when $k \geqslant 3$.

A graph G is factor-critical if $G-v$ has a perfect matching for every vertex $v \in V(G)$ and is bicritical if $G-u-v$ has a perfect matching for every pair of distinct vertices $u, v \in V(G)$. More generally, a graph is said to be k-factor-critical if $G-S$ has a perfect matching for every set S of k vertices in G. In three previous papers [N. Ananchuen, M.D. Plummer, Some results related to the toughness of 3-domination-critical graphs, Discrete Math. 272 (2003) 5–15; N. Ananchuen, M.D. Plummer, Matching properties in domination critical graphs, Discrete Math. 277 (2004) 1–13; N. Ananchuen, M.D. Plummer, Some results related to the toughness of 3-domination-critical graphs. II. Utilitas Math. 70 (2006) 11–32], we explored the toughness of 3-γ-critical graphs and some of their matching properties. In particular, we obtained some properties which are sufficient for a 3-γ-critical graph to be factor-critical and, respectively, bicritical. In the present work, we obtain similar results for k-factor-critical graphs when $k=3$.

Keywords: Domination; Critical edge; k-γ-Critical; Matching; Factor-critical; Bicritical; k-Factor-critical

1. Introduction

Let G denote a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$. The minimum degree of all vertices in G will be denoted by $\delta(G)$. A set $S \subseteq V(G)$ is a *dominating set* for G if every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum cardinality of a dominating set in graph G is called the *domination number* of G and is denoted by $\gamma(G)$. Graph G is said to be *k-γ-critical* if $\gamma(G)=k$, but $\gamma(G+e)=k-1$ for each edge $e \notin E(G)$. In this paper, we will be concerned only with the case $k=3$.

If u, v and w are vertices of G and u and v dominate $G-w$, we will follow previously accepted notation and write $[u, v] \longrightarrow w$. Suppose G is 3-γ-critical. If u and v are non-adjacent vertices of G, then $\gamma(G+uv)=2$ and so there is a vertex $x \in V(G)$ such that either $[u, x] \longrightarrow v$ or $[v, x] \longrightarrow u$.

Sumner and Blitch [11] initiated work on matchings in 3-γ-critical graphs and the following lemma of theirs will prove very useful in our work to follow. A complete proof may be found in [11] together with [8].

E-mail addresses: nawarat@kanate.su.ac.th (N. Ananchuen), michael.d.plummer@vanderbilt.edu (M.D. Plummer).
[1] Work supported by the Thailand Research Fund Grant #BRG4380016.
[2] Work supported by NSF Grant # INT-9816113.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.03.012

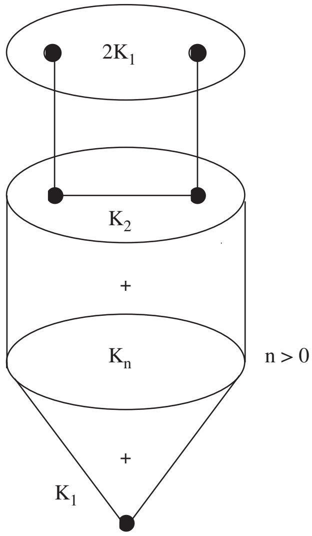

Fig. 1.

Lemma 1.1. *Let G be a connected* 3-γ-*critical graph and let S be an independent set of* $n \geqslant 2$ *vertices in* $V(G)$.

(i) Then the vertices of S can be ordered $a_1, a_2, \ldots, a_n$ in such a way that there exists a sequence of distinct vertices $x_1, x_2, \ldots, x_{n-1}$ so that $[a_i, x_i] \longrightarrow a_{i+1}$ for $i = 1, 2, \ldots, n-1$.
(ii) If, in addition, $n \geqslant 4$, then the x_i's can be chosen so that $x_1 x_2 \cdots x_{n-1}$ is a path and $S \cap \{x_1, \ldots, x_{n-1}\} = \emptyset$.

In what is to follow, we shall also make frequent use of the following easy result.

Lemma 1.2. *Let G be a* 3-γ-*critical graph and let u and v be non-adjacent vertices of G. If x is a vertex of G such that* $[u, x] \longrightarrow v$, *then* $xv \notin E(G)$ *and if x is a vertex of G with* $[v, x] \longrightarrow u$ *then* $xu \notin E(G)$.

In [1] the following result was obtained. (See also [5].)

Theorem 1.3. *Let G be a connected* 3-γ-*critical graph and let S be a vertex cutset in G. Then*

(i) *if* $|S| \geqslant 4$, $G - S$ *has at most* $|S| - 1$ *components*,
(ii) *if* $|S| = 3$, *then* $G - S$ *contains at most* $|S|$ *components*, *and if* $G - S$ *has exactly three components*, *then each component is complete and at least one is a singleton*,
(iii) *if* $|S| = 2$, *then* $G - S$ *has at most three components and if* $G - S$ *has exactly three components*, *then G must have the structure shown in Fig*. 1 *and*,
(iv) *if* $|S| = 1$, *then* $G - S$ *has two components*, *exactly one of which is a singleton. Furthermore*, *in case* (iv), *G has at most three cutvertices. If it has three*, *G is isomorphic to the graph shown in* Fig. 1 *with* $n = 1$. *If it has two*, *G is isomorphic to a graph of the family shown in Fig*. 1 *with* $n \geqslant 2$.

We refer the reader to [9] for further notation, terminology and background for matching theory. In particular, $N(v)$ will denote the neighborhood of vertex v, that is, the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ will denote the closed neighborhood of v. In addition, we denote by $\omega(G)$ the number of components of the graph G and by $\omega_o(G)$, the number of components of odd order in G.

In order to prove our main results, we shall need the following two theorems from [3], both of which may be viewed as extensions of Theorem 1.3.

Theorem 1.4. *If G is a connected* 3-γ-*critical graph and S is a vertex cutset in G*, *then if* $|S| \geqslant 6$, *it follows that* $\omega(G - S) \leqslant |S| - 2$.