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Choosability of graphs with infinite sets of forbidden differences☆

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Abstract

The notion of the list-T-coloring is a common generalization of the T-coloring and the list-coloring. Given a set of non-negative integers T, a graph G and a list-assignment L, the graph G is said to be T-colorable from the list-assignment L if there exists a coloring c such that the color $c(v)$ of each vertex v is contained in its list $L(v)$ and $|c(u) - c(v)| \notin T$ for any two adjacent vertices u and v. The T-choice number of a graph G is the minimum integer k such that G is T-colorable for any list-assignment L which assigns each vertex of G a list of at least k colors.

We focus on list-T-colorings with infinite sets T. In particular, we show that for any fixed set T of integers, all graphs have finite T-choice number if and only if the T-choice number of K_2 is finite. For the case when the T-choice number of K_2 is finite, two upper bounds on the T-choice number of a graph G are provided: one being polynomial in the maximum degree of the graph G, and the other being polynomial in the T-choice number of K_2.
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1. Introduction

Special types of graph colorings attracted attention of researchers in connection with their applications in wireless networks. Hale [7] formulated several frequency assignment problems in the terms of graph theory. Suppose that transmitters are stationed at various locations, and we wish to assign every transmitter a frequency over which it will operate. The frequencies need to be assigned in a way such that the frequencies of nearby standing transmitters do not interfere. If interference occurred only when the transmitters use the same frequency, the problem could be formulated as a graph-coloring problem: every transmitter is represented by a vertex, and frequencies are referred to as colors. Any pair of vertices representing close transmitters is connected by an edge.

In practice, interference occurs even if the frequencies are different, e.g., when the difference of the frequencies equals a certain value. T-colorings of graphs deal with this restriction: given a set of nonnegative integers T, a T-coloring of a graph G is a vertex-coloring (with positive integers) of G such that the absolute value of the difference between any two colors assigned to adjacent vertices does not belong to the set T. Let us remark that the set $T = \{0, 7, 14, 15\}$ is the set of forbidden differences in the model for frequency assignment in UHF television transmitter systems [9]. The

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concept of T-colorings has been extensively studied as witnessed by the survey of Roberts [10]. The reader is referred to this survey for more detailed introduction.

However, it is not always possible for a given transmitter to operate on all frequencies—instead a transmitter is assigned a set of frequencies over which it can operate. This leads us to the concept of *list colorings* introduced independently by Vizing [14], and by Erdős et al. [4]. Combining list colorings with T-colorings, *list-T-colorings* arise. This notion was first introduced by Tesman [12] and further studied by Alon and Zaks [2], Fiala et al. [5], Tesman [11], Waller [15,16], and others. Given a set $L(v)$ of allowed colors for each vertex of a graph G, a list-T-coloring of G is a proper T-coloring of the graph G such that the color assigned to a vertex v belongs to the set $L(v)$. A graph G is said to be *T-k-choosable* if a list-T-coloring exists for every collection of sets $L(v)$ such that $|L(v)| = k$ for every vertex v. The *T-choice number* $\mathrm{ch}_T(\mathrm{G})$ of a graph G is the minimum number k such that G is T-k-choosable.

So far, researchers mainly focused on the case when the set T is finite. Alon and Zaks [2] proposed to consider the case when T is infinite, in particular, to characterize those infinite sets T for which the T-choice numbers are finite for all graphs. In this paper, we attempt to make the first step in this direction. In particular, it is shown that for any set T, the T-choice number is either finite for all graphs or infinite for all (nonempty) graphs.

We also investigate the behavior of the T-choice number of a given graph G in terms of its maximum degree Δ. In order to do this, we introduce the following function:

Definition 1. If T is a set of integers, then $\mathrm{ch}_T(\Delta)$ is the smallest integer ℓ such that every graph with maximum degree Δ is T-ℓ-choosable.

In Section 3, we prove that for any integer $\Delta \geqslant 1$, $\mathrm{ch}_T(\Delta)$ is finite if and only if $\mathrm{ch}_T(1)$ is finite. For the case when the $\mathrm{ch}_T(1)$ is finite, two upper bounds on the T-choice number of a graph G are provided in Sections 3 and 4: one being polynomial in the maximum degree of the graph G and the other polynomial in $\mathrm{ch}_T(1) = \mathrm{ch}_T(K_2)$. At the end of the paper, we investigate the connection between $\mathrm{ch}_T(\Delta)$ and the length of the longest arithmetical progression contained in T.

2. Preliminaries

Throughout the paper, the following notation is used: if a is an integer and B is a set of integers, then $a + B$ denotes the set $\{a + b : b \in B\}$. If A and B are two sets of integers, $A + B$ denotes the set $\{a + b : a \in A, b \in B\}$. Similarly, $-A$ stands for the set $\{-a : a \in A\}$. Two colors c_1 and c_2 are said to be *conflicting with respect to a set T* if $|c_1 - c_2| \in T$. When the set T is clear from the context, the colors are said just to be *conflicting*. A *nonempty* graph is a graph that contains at least one edge.

Next, we establish a proposition which outlines the connection between T-choice number of the graph K_2 (i.e., a single edge) and structure of the set T.

Proposition 2. *Let $k \geqslant 2$ be an integer. The graph K_2* is *T-k-choosable if and only if the following inequality holds for every k distinct integers $i_1, \ldots, i_k$:*

$$|(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))| < k.$$

Proof. Let u and v be the two vertices of K_2. Firstly, consider the case when there exist k distinct integers $i_1, \ldots, i_k$ such that

$$|(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))| \geqslant k.$$

Let $L(u)$ be $\{i_1, \ldots, i_k\}$ and $L(v)$ be any k-element subset of the set $(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))$. Now, it is impossible to color properly both u and v from their lists because all colors in $L(u)$ conflict with all colors in $L(v)$.

The other implication is also not too difficult: fix a list-assignment L, and let $L(u) = \{i_1, \ldots, i_k\}$. Because

$$|(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))| < k,$$

there exist colors $c_1 \in L(u)$ and $c_2 \in L(v)$ such that $|c_1 - c_2| \notin T$. Otherwise, we have that the set $L(v)$ is contained in the above intersection, so the size of the intersection must be at least k, a contradiction. Now, we can use c_1 to color u and c_2 to color v and we obtain a proper coloring. Hence, K_2 can be colored properly from the list-assignment L. □

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