Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 175 – 179
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

# Primitive 2-factorizations of the complete graph

Giuseppe Mazzuoccolo[1]

*Dipartimento di Matematica, Università di Modena e Reggio Emilia, via Campi 213/B, 41100 Modena, Italy*

Received 30 November 2004; received in revised form 15 October 2005; accepted 2 February 2006
Available online 2 June 2007

**Abstract**

Let $\mathscr{F}$ be a 2-factorization of the complete graph $K_v$ admitting an automorphism group $G$ acting primitively on the set of vertices. If $\mathscr{F}$ consists of Hamiltonian cycles, then $\mathscr{F}$ is the unique, up to isomorphisms, 2-factorization of $K_{p^n}$ admitting an automorphism group which acts 2-transitively on the vertex-set, see [A. Bonisoli, M. Buratti, G. Mazzuoccolo, Doubly transitive 2-factorizations, J. Combin. Designs 15 (2007) 120–132.]. In the non-Hamiltonian case we construct an infinite family of examples whose automorphism group does not contain a subgroup acting 2-transitively on vertices.


*MSC:* 05C70; 05C15; 05C25

*Keywords:* Factorization; Coloring of graphs and hypergraphs; Graphs and groups

## 1. Introduction

For an integer $v \geqslant 3$, let $K_v$ be the complete (simple undirected) graph on $v$ vertices with vertex-set $V(K_v)$ and edge set $E(K_v)$. For $3 \leqslant k \leqslant v$, a *k-cycle* $C = (x_0, x_1, \ldots, x_{k-1})$ is the subgraph of $K_v$ whose edges are $[x_i, x_{i+1}]$, $i = 0, \ldots, k-1$, indices taken modulo $k$. If $k = v$, the cycle is called Hamiltonian.

A 2-*factor* $F$ of $K_v$ is a set of cycles whose vertices partition $V(K_v)$. A 2-*factorization* of $K_v$ is a set $\mathscr{F}$ of edge disjoint 2-factors forming a cover of $E(K_v)$. A 2-factorization in which all the 2-factors are isomorphic to a factor $F$ is called an *F-factorization*. If each 2-factor of $\mathscr{F}$ consists of a single Hamiltonian cycle, $\mathscr{F}$ is called a *Hamiltonian 2-factorization*. The existence of a 2-factorization of $K_v$ forces $v$ to be odd.

The collection of cycles appearing in the factors of $\mathscr{F}$ form a cycle decomposition of $K_v$, which is called the underlying decomposition. We will denote it by $\mathscr{D}_{\mathscr{F}}$.

A permutation group $G$ acting faithfully on $V(K_v)$ and preserving the 2-factorization $\mathscr{F}$ is called an *automorphism group* of $\mathscr{F}$.

In some recent papers the possible structures and actions of $G$ on vertices or factors have been investigated. In [3] the situation in which $G$ acts regularly (i.e. sharply transitively) on vertices is studied in detail. In [2] a complete description of $G$ and $\mathscr{F}$ is given in case the action of $G$ is doubly transitive on the vertex-set. In particular it is proved that if $\mathscr{F}$ is Hamiltonian, then $v$ is an odd prime $p$, the group $G$ is the affine general linear group $\mathrm{AGL}(1, p)$ and, if the vertices of

---

[1] Research performed with the financial support of the Italian Ministry MIUR, project “Strutture Geometriche, Combinatoria e loro Applicazioni”.
*E-mail address:* mazzuoccolo@unimore.it.

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2006.02.017

$K_p$ are labelled by the elements of $\mathbb{Z}_p$, then $\mathscr{F} = \{C_1, C_2, \ldots, C_{(p-1)/2}\}$, with $C_i = (0, i, 2i, \ldots, (p-1)i)$ (subscripts mod $p$), $i = 1, 2, \ldots, (p-1)/2$. This factorization is the *natural* 2-factorization (also denoted by $\mathscr{N}(\mathbb{Z}_p)$) which arises from the cyclic group $\mathbb{Z}_p$, see [3].

In this paper, we investigate primitive 2-factorizations, i.e. admitting an automorphism group $G$ with primitive action on the vertex-set. Note that all 2-factorizations admitting an automorphism group doubly transitive on the vertex-set are also examples of primitive 2-factorizations.

If $\mathscr{F}$ is Hamiltonian, we prove that $v$ is an odd prime $p$ and $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$. Moreover, the group $G$ is necessarily a subgroup of $\mathrm{AGL}(1, p)$ containing $\mathbb{Z}_p$. In the non-Hamiltonian case, we give examples of primitive 2-factorizations whose full automorphism group does not contain a subgroup acting doubly transitively on the vertices. In the last section we also prove that a primitive 2-factorization of $K_9$ is necessarily 2-transitive, whence a 2-factorization arising from the affine line parallelism of $\mathrm{AG}(2, 3)$ in a suitable manner, see [2], and a primitive 2-factorization of $K_{15}$ does not exist.

## 2. The Hamiltonian case

In this section we prove that $\mathscr{N}(\mathbb{Z}_p)$ is the unique primitive Hamiltonian 2-factorization of a complete graph.

**Lemma 1.** *Let $\mathscr{F}$ be a 2-factorization of $K_p$ with a transitive automorphism group $G$. Then $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$ and $G \leqslant \mathrm{AGL}(1, p)$.*

**Proof.** By transitivity of $G$ on $V(K_p)$, the integer $p$ is a divisor of the order of $G$, then an element $g \in G$ of order $p$ exists. The cyclic group generated by $g$ acts regularly on the vertex-set. By proposition 2.8 of [3], it is $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$. The full group of automorphism of $\mathscr{F}$ is $\mathrm{AGL}(1, p)$ (see [2], Section 1) and then $G \leqslant \mathrm{AGL}(1, p)$. $\square$

**Theorem 1.** *Let $\mathscr{F}$ be a Hamiltonian 2-factorization of $K_v$ with primitive automorphism group $G$. Then $v = p$, $\mathscr{F} = \mathscr{N}(\mathbb{Z}_p)$ and $\mathbb{Z}_p \leqslant G \leqslant \mathrm{AGL}(1, p)$.*

**Proof.** Suppose $G$ is of even order. We prove that $G$ contains exactly $v$ involutions. First of all observe that each involution of $G$ fixes all the 2-factors of $\mathscr{F}$. In fact let $g \in G$ be an involution exchanging two vertices $x_0$ and $x_1$. Labelling the vertices such that $C = (x_0, x_1, \ldots, x_{v-1})$ is the unique cycle of $\mathscr{D}_{\mathscr{F}}$ containing $[x_0, x_1]$, we obtain:

$$g(x_i) = x_{v+1-i}, \quad g(x_{v+1-i}) = x_i \quad \text{for } i = 1, \ldots, \frac{v-1}{2},$$
$$g(x_{(v+1)/2}) = x_{(v+1)/2},$$

where all indices are taken mod $v$. Then $g$ fixes the vertex $x_{(v+1)/2}$ and each edge of the set $E = \{[x_i, x_{v+1-i}]/i = 1, \ldots, (v-1)/2\}$. Suppose that at least two edges of $E$ belong to the same 2-factor $F$ of $\mathscr{F}$. Then $g$ should fix the unique cycle of $F$ and two edges on it: a contradiction. We have proved that the elements in $E$ are in different 2-factors. By the fact that the cardinality of $E$ coincides with the number of 2-factors, we conclude that $g$ fixes all the factors of $\mathscr{F}$. Furthermore we can also observe that each involution in $G$ fixes exactly one vertex of $V(K_v)$. Let now $x \in V(K_v)$, we have $|G| = |G_x|v$, where $G_x$ is the stabilizer of the vertex $x$, then $|G_x|$ is even and we have at least one involution in $G_x$. Observe that $G_x$ contains exactly one involution; in fact if we fix a 2-factor $F$ and $C$ is its cycle, the action of an involution of $G_x$ is uniquely determined by its action on the vertices of $C$ as above explained. We can conclude that the group $G$ contains exactly $v$ distinct involutions. In particular for each factor $F$, the subgroup $G_F$ contains $v$ involutions: namely all the involutions of $G$. It is well known that $G_F \leqslant D_v$, the dihedral group on $v$ vertices, and then $G_F \cong D_v$ for each $F \in \mathscr{F}$. Furthermore, for each factor $F' \in \mathscr{F} - \{F\}$, the dihedral groups $G_F$ and $G'_F$ contain exactly the same $v$ involutions, therefore $G_F = G'_F$. Label the vertices of $K_v$ by the elements $0, 1, \ldots, v-1$ in such a way that a 2-factor $F \in \mathscr{F}$ contains the cycle $(0, 1, \ldots, v-1)$and an element of $G_F$ of order $v$ maps the vertex $i$ onto $i+1$, for each $i$. Denote by $g \in G_F$ this element. Suppose $v$ is not a prime and let $h$ be a proper divisor of $v$. Let $F' \in \mathscr{F}$ be the 2-factor containing the edge $[0, h]$. As $G_{F'} = G_F$ we have $g^h \in G_{F'}$ and then $F'$ contains the cycle $(0, h, 2h, \ldots, v-h)$ of length less than $v$: a contradiction. We conclude that $v$ is a prime. By Lemma 1 the assertion follows in this case. We have proved that for a primitive Hamiltonian 2-factorization of $K_v$ only two possibilities hold: either $v$ is a prime or $|G|$ is odd. By the O'Nan Scott Theorem (see [6]) and by the fact that a simple non-abelian group