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Spaces with congruence

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Abstract

In this paper we consider an exchange space with congruence $(P, \mathfrak{L}, \equiv)$ not assuming any further geometrical properties. If the dimension of the space is greater than 2, we show that for any line G of a plane E and any point $x \in G$ there is a unique perpendicular line through x in E and that any line reflection is a motion.

Keywords: Congruence; Motions; Line reflections

1. Introduction

It is well known that every Euclidean plane $(E, \mathfrak{L}, \alpha, \equiv)$ [2,9] is isomorphic to an affine plane AG(2, K) over an ordered and pythagorean commutative field K (cf. [2] (24.5)). Here $(E, \mathfrak{L})$ denotes an affine plane, $(E, \mathfrak{L}, \alpha)$ an ordered plane, and $\equiv$ denotes the congruence relation on $E \times E$. We may consider E as a quadratic separable field extension of K with the corresponding involutory field automorphism $^{-} : E \to E$ and we have $(a, b) \equiv (c, d)$ if and only if $(a - b)\overline{(a - b)} = (c - d)\overline{(c - d)}$ for any points $a, b, c, d \in E$.

There is a corresponding theorem for hyperbolic planes (cf. [2]). For both proofs one first considers the group of motions, in particular the line reflections. For the definition of a motion and a line reflection the order relation is not necessary. We need only the linear structure of $(E, \mathfrak{L})$ and the congruence relation $\equiv$, but for the proof that the line reflection is a motion it seems that additional assumptions on the geometry are necessary. For example Sörensen assumes in [8] that for given lines G_1, G_2, there exist distinct lines H_1, H_2 through a common point z which intersect G_1, G_2. Here we give a proof for exchange planes with congruence not using this property.

In a plane with congruence $(E, \mathfrak{L}, \equiv)$ two perpendicular lines may have an empty intersection. We show that if in a plane any two perpendicular lines have an empty intersection, then the relation "perpendicular" together with the "identity relation" is transitive (cf. Theorem 2.13).

If there are two perpendicular lines with a non-empty intersection, one can show that for any line G and any point $x \in G$ we have a perpendicular line through x. But it is open if there is a unique perpendicular line to G through x. This is true for finite or affine planes, but not known in general. In this paper we consider a space with congruence $(P, \mathfrak{L}, \equiv)$ not assuming any further geometrical properties. (For special spaces with congruence cf. [5,6].) We only assume that the space satisfies the exchange property, which for example can be easily shown for ordered spaces. In Section 3 we assume that there are two points with a midpoint. We use the property that if two points have a midpoint,

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then they have a midline in every plane. If the dimension of the space is greater than 2, we show that for any line G of a plane E and any point $x \in G$ there is a unique perpendicular line through x in E and that any line reflection is a motion. It follows that for any two points b, z there exists a unique point $b' \in \overline{z, b} \setminus \{b\}$ with $(z, b) \equiv (z, b')$ and also point reflections are motions. For the case that there exist no points with a midpoint, we conjecture that all examples in which the line reflections are motions have dimension 2.

2. Spaces with congruence

Let $(P, \mathfrak{L})$ denote a *linear space* or *incidence space* with the point set P, the line set $\mathfrak{L}$ and at least three points on every line, i.e.,

- for any two points there is exactly one line containing them and
- for any line $L \in \mathfrak{L}$ we have $|L| \geqslant 3$.

A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line passing through x and y, denoted by $\overline{x, y}$, is contained in U. Let $\mathfrak{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operation*

$$\overline{} : \mathfrak{P}(P) \to \mathfrak{U};\ X \mapsto \overline{X} \quad \text{by} \quad \overline{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U. \tag{1}$$

For $U \in \mathfrak{U}$ we call $\dim U := \inf\{|X| - 1 : X \subset U \text{ and } \overline{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*. For a set $\{a, b, c, \ldots\}$ we write $\overline{a, b, c, \ldots}$ instead of $\overline{\{a, b, c, \ldots\}}$.

We introduce the concept of a space $(P, \mathfrak{L}, \equiv)$ with congruence (cf. [8]). We assume that $(P, \mathfrak{L})$ is a linear space which satisfies the following exchange condition.

(EC) Let $S \subset P$ and let $x, y \in P$ with $x \in \overline{S \cup \{y\}} \setminus \overline{S}$. Then $y \in \overline{S \cup \{x\}}$

Let $\equiv$ be a *congruence relation* on $P \times P$, i.e., $\equiv$ is an equivalence relation with $(a, b) \equiv (b, a)$, $(a, a) \equiv (b, b)$ and $(a, a) \equiv (b, c)$ implies $b = c$.

We use the notation $(x_1, x_2, x_3) \equiv (y_1, y_2, y_3)$ if and only if $(x_i, x_j) \equiv (y_i, y_j)$ for $i, j \in \{1, 2, 3\}$. $(P, \mathfrak{L}, \equiv)$ is a *space with congruence* if the axioms (W1), (W2) and (W3) are satisfied.

(W1) Let $a, b, c \in P$ be distinct and collinear, and let $a', b' \in P$ with $(a, b) \equiv (a', b')$. Then there exists exactly one $c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$.

(W2) Let $a, b, x \in P$ be non-collinear and let $a', b', x' \in P$ with $(a, b, x) \equiv (a', b', x')$. For any $c \in \overline{a, b}$ and $c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$ it holds $(x, c) \equiv (x', c')$.

(W3) For $a, b, x \in P$ non-collinear there exists exactly one $x' \in \overline{a, b, x} \setminus \{x\}$ with $(a, b, x) \equiv (a, b, x')$.

We call a bijective mapping $\phi : P \to P$ a *motion*, if $(x, y) \equiv (\phi(x), \phi(y))$ for all $x, y \in P$.

Lemma 2.1. (i) *If a, b, c are collinear points and $a', b', c' \in P$ with $(a, b, c) \equiv (a', b', c')$, then a', b', c' are collinear.*
(ii) *Any motion ϕ is a collineation.*

Proof. (i) By (W1) the point $c'' \in \overline{a', b'}$ exists with $(a', b', c'') \equiv (a, b, c)$. If $c' \notin \overline{a', b'}$, then by (W2) it would follow $(c', c'') \equiv (c, c)$, hence $c' = c'' \in \overline{a', b'}$.
(ii) By (i), ϕ and ϕ^{-1} map collinear points onto collinear points. $\square$

For a subspace U and points $a, b \in U$ we define $M_U(a, b) := \{x \in U : (a, x) \equiv (b, x)\}$. We call $M_U(a, b)$ a *midpoint*, a *midline*, or a *midplane* of a, b, respectively, if it is a point, a line, or a plane, respectively.

Lemma 2.2. (i) *Two distinct points a, b have at most one point $m \in \overline{a, b}$ with $(a, m) \equiv (b, m)$.*
(ii) *$M_U(a, b)$ is a subspace of U.*