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Double graphs☆

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Abstract

In this paper we study the elementary properties of double graphs, i.e. of graphs which are the direct product of a simple graph G with the graph obtained by the complete graph K_2 adding a loop to each vertex.

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1. Introduction

In [18] it was observed that the binary strings of length $n+1$ without zigzags, i.e. without 010 and 101 as factors, can be reduced to the Fibonacci strings, i.e. binary strings without two consecutive 1's, of length n. The set of Fibonacci strings can be endowed with a graph structure saying that two strings are adjacent when they differ exactly in one position. The graphs obtained in this way are called *Fibonacci cubes* [12] and have been studied in several recent papers. We wondered if the set of all binary strings without zigzags could be endowed with some graph structure related in some way with Fibonacci cubes. One interesting such graph structure is the one induced by the graph structure of Fibonacci strings, that is the one obtained defining the adjacency saying that two binary strings without zigzags are adjacent if and only if the corresponding Fibonacci strings are adjacent as vertices of the Fibonacci cube. The resulting graph can be build up taking two distinct copies of the Fibonacci cube Γ_n and joining every vertex v in one component to every vertex w' in the other component corresponding to a vertex w adjacent to v in the first component. At this point it was straightforward to observe that this is a general construction which can be performed on every simple graph. We called *double graphs* all the graphs which can be obtained in such a way. Since the class of double graphs with this construction turned out to have several interesting properties, we decided to write this paper as an elementary introduction to such graphs that perhaps deserve to be better known.

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2. Definitions

In this paper we will consider only finite simple graphs (i.e. without loops and multiple edges). As usual $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively, and adj denote the adjacency relation of G. For all definitions not given here see [1,5,9,13,17].

The *direct product* of two graphs G and H is the graph $G \times H$ with $V(G \times H) = V(G) \times V(H)$ and with adjacency defined by (v_1, w_1) adj (v_2, w_2) if and only if v_1 adj v_2 in G and w_1 adj w_2 in H.

The *total graph* T_n on n vertices is the graph associated to the total relation (where every vertex is adjacent to every vertex). It can be obtained from the complete graph K_n by adding a loop to every vertex. In [13] it is denoted by K_n^s.

We define the *double* of a simple graph G as the graph $\mathscr{D}[G] = G \times T_2$. Since the direct product of a simple graph with any graph is always a simple graph, it follows that the double of a simple graph is still a simple graph.

In $\mathscr{D}[G]$ we have $(v, h)\,\mathrm{adj}(w, k)$ if and only if v adj w in G. Then, if $V(T_2) = \{0, 1\}$, we have that $G_0 = \{(v, 0) : v \in V(G)\}$ and $G_1 = \{(v, 1) : v \in V(G)\}$ are two subgraphs of $\mathscr{D}[G]$ both isomorphic to G such that $G_0 \cap G_1 = \emptyset$ and $G_0 \cup G_1$ is a spanning subgraph of $\mathscr{D}[G]$. Moreover we have an edge between $(v, 0)$ and $(w, 1)$ and similarly we have an edge between $(v, 1)$ and $(w, 0)$ whenever v adj w in G. We will call $\{G_0, G_1\}$ the *canonical decomposition* of $\mathscr{D}[G]$. See Fig. 1 for some examples.

From the above observations it follows that if G has n vertices and m edges then $\mathscr{D}[G]$ has $2n$ vertices and $4m$ edges. In particular $\deg_{\mathscr{D}[G]}(v, k) = 2 \deg_G(v)$.

The *lexicographic product* (or *composition*) of two graphs G and H is the graph $G \circ H$ with $V(G) \times V(H)$ as vertex set and with adjacency defined by $(v_1, w_1)\,\mathrm{adj}(v_2, w_2)$ if and only if $v_1 = v_2$ and w_1 adj w_2 in H or v_1 adj v_2 in G. The graph $G \circ H$ can be obtained from G substituting to each vertex v of G a copy H_v of H and joining every vertex of H_v with every vertex of H_w whenever v and w are adjacent in G [13, p. 185].

Lemma 1. *For any graph G we have $G \times T_n = G \circ N_n$, where N_n is the graph on n vertices without edges.*

Proof. For simplicity consider T_n and N_n on the same vertex set. Then the function $f : G \times T_n \to G \circ N_n$, defined by $f(v, k) = (v, k)$ for every $(v, k) \in V(G \times T_n)$, is a graph isomorphism. Indeed, since N_n has no edges, we have that (v, h) adj (w, k) in $G \circ N_n$ if and only if v adj w in G. □

From Lemma 1 it immediately follows that:

Proposition 2. *For any graph G on n vertices, $\mathscr{D}[G] = G \circ N_2$ and $\mathscr{D}[G]$ is n-partite* (*Fig. 2*).

We will write $\mathscr{D}^2[G]$ for the double of the double of G. More generally we will have the graphs $\mathscr{D}^k[G] = G \times T_{2^k} = G \circ N_{2^k}$, for every $k \in \mathbb{N}$.

The given definition of double graph can be generalized considering the operator $\mathscr{D}_k$ defined by $\mathscr{D}_k[G] = G \times T_k$ for every simple graph G. For Lemma 1 it is also $\mathscr{D}_k[G] = G \circ N_k$ for every simple graph G. Moreover the powers of

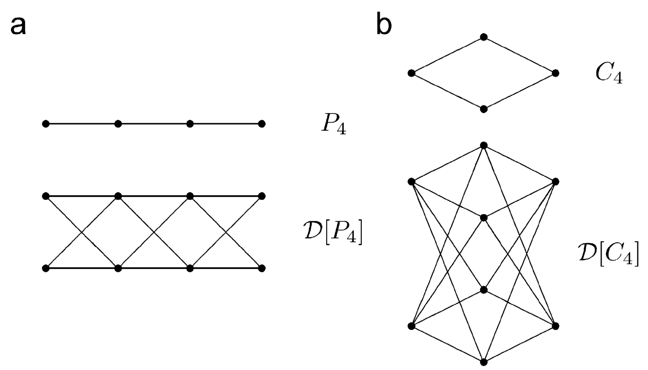

Fig. 1. (a) A path and its double, (b) a cycle and its double.