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# The bondage numbers of graphs with small crossing numbers☆

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**Abstract**

The bondage number $b(G)$ of a nonempty graph $G$ is the cardinality of a smallest edge set whose removal from $G$ results in a graph with domination number greater than the domination number $\gamma(G)$ of $G$. Kang and Yuan proved $b(G) \leqslant 8$ for every connected planar graph $G$. Fischermann, Rautenbach and Volkmann obtained some further results for connected planar graphs. In this paper, we generalize their results to connected graphs with small crossing numbers.


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*Keywords:* Bondage number; Domination number; Crossing number; Planar graph

## 1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [7]. Let $G = (V, E)$ be a finite, undirected and simple graph. For each vertex $u \in V(G)$, let $N_G(u)$ be the neighborhood of $u$ and $N_G(X) = \cup_{x \in X} N_G(x)$. We denote the degree of $u$ by $d_G(u) = |N_G(u)|$, the maximum and the minimum degree of $G$ by $\Delta(G)$ and $\delta(G)$, respectively, and the distance between the vertices $x$ and $y$ by $d_G(x, y)$. Let $n_i = n_i(G)$ be the number of vertices of degree $i$ for $i = 1, 2, \ldots, \Delta(G)$. The girth of $G$, $g(G)$, is the length of the shortest cycle in $G$. If $G$ has no cycles we define $g(G) = \infty$. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$. The *crossing number* of $G$, $\mathrm{cr}(G)$, is the smallest number of pairwise intersections of its edges when $G$ is drawn in the plane. If $\mathrm{cr}(G) = 0$, then $G$ is a planar graph.

A subset $D$ of $V(G)$ is called a *dominating set*, if $D \cup N(D) = V(G)$. The minimum cardinality of all dominating sets in $G$ is called the *domination number*, and denoted by $\gamma(G)$. The *bondage number* of a nonempty graph $G$, $b(G)$, is the cardinality of a minimum set of edges whose removal from $G$ results in a graph with domination number larger than $\gamma(G)$.

The first result on bondage numbers was obtained by Bauer et al. [1]. Dunbar et al. [2] conjectured that $b(G) \leqslant \Delta(G) + 1$ for any nontrivial planar graph $G$. Kang and Yuan [5] confirmed this conjecture for $\Delta(G) \geqslant 7$ by proving that $b(G) \leqslant \min\{8, \Delta(G) + 2\}$, and proved that $b(G) \leqslant 7$ for any connected planar graph without vertices of degree five. Fischermann et al. [3] generalized the latter result, and showed that the conjecture is valid for all connected planar

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graphs with $g(G) \geqslant 4$ and $\Delta(G) \geqslant 5$ as well as all planar graphs with $g(G) \geqslant 5$ unless they are 3-regular. We generalize these results to connected graphs with small crossing numbers.

The rest of the paper is organized as follows. In the next section, we recall some results to be used in our discussions. Our main results are given in Sections 3 and 4. In Section 3, we discuss the upper bound of $b(G)$ for a connected graph $G$ with $g(G) \geqslant 4$. In Section 4, we discuss the upper bound of $b(G)$ for connected graph $G$ with some degree constraints.

## 2. Some lemmas

In this section, we recall some useful known results on the bondage number.

**Lemma 2.1** (*Bauer et al.* [1], *Teschner* [6]). *If $G$ is a nontrivial graph, then $b(G) \leqslant d_G(u) + d_G(v) - 1$ for any two distinct vertices $u$ and $v$ with $d_G(u, v) \leqslant 2$ in $G$.*

**Lemma 2.2** (*Bauer et al.* [1]). *Let $G$ be a graph with $\delta(G) \geqslant 1$. Then $b(G) \leqslant 2$ if $G$ is a tree, and $b(G) \leqslant \Delta(G) + \delta(G) - 1$ otherwise.*

**Lemma 2.3** (*Hartnell and Rall* [4], *Teschner* [6]). *If $G$ has edge-connectivity $\lambda(G) \geqslant 1$, then $b(G) \leqslant \Delta(G) + \lambda(G) - 1$.*

**Lemma 2.4** (*Hartnell and Rall* [4]). *If $G$ is a nontrivial graph, then $b(G) \leqslant d_G(u) + d_G(v) - 1 - |N_G(u) \cap N_G(v)|$ for any adjacent vertices $u$ and $v$ in $G$.*

The following two results about planar graphs are well-known (cf. [7]).

**Lemma 2.5** (*Euler's Formula*). *If $G$ is a planar graph with $n(G)$ vertices, $m(G)$ edges, $\omega(G)$ components and $\phi(G)$ regions, then $\phi(G) = m(G) - n(G) + \omega(G) + 1$.*

**Lemma 2.6.** *For a planar graph $G$, $m(G) \leqslant 3n(G) - 6$ if $n(G) \geqslant 3$ and $m(G) \leqslant 2n(G) - 4$ if $G$ is bipartite and $n(G) \geqslant 3$.*

**Lemma 2.7** (*Fischermann et al.* [3]). *If $G$ is a planar graph with $3 \leqslant g(G) < \infty$ and the number $c(G)$ of cut-edges, then*

$$m(G) \leqslant \frac{g(G)(n(G) - 2) - c(G)}{g(G) - 2}.$$

Let $F_1$ be the graph with the vertex-set $\{u, u_1, u_2, u_3\}$ and the edge-set $\{uu_i |\ i = 1, 2, 3\} \cup \{u_1u_2\}$ and $F_2$ be the graph with the vertex-set $\{v, v_1, v_2, v_3, v_4\}$ and the edge-set $\{vv_i |\ i = 1, 2, 3, 4\} \cup \{v_1v_2, v_3v_4\}$. Furthermore, for every positive integer $t$, let $H_{2,t}$ be the graph obtained from the complete bipartite graph $K_{2,t}$ with the partite sets $\{x, y\}$ and $\{w_1, w_2, \ldots, w_t\}$ by adding an edge $xy$. Now we define $\mathscr{G} = \{C_4, C_5, F_1, F_2\} \cup \{H_{2,t} |\ t \geqslant 1\}$, where $C_4$ and $C_5$ are cycles of length 4 and 5, respectively.

**Lemma 2.8** (*Fischermann et al.* [3]). *Let $G$ be a connected planar graph with $3 \leqslant g(G) < \infty$. Then $G \notin \mathscr{G}$ if and only if*

$$3\phi(G) \leqslant 2m(G) - n_2(G) - n_1(G).$$

**Lemma 2.9** (*Kang and Yuan* [5]). *$G$ is a planar graph and $v \in V(G)$ with $d_G(v) \geqslant 2$. Let $E_v = \{xy |\ x, y \in N_G(v)$ and $xy \notin E(G)\}$. Then there is a subset $F \subseteq E_v$ such that $H = G + F$ is still a planar graph and $H[N_G(v)]$ is 2-connected when $d_G(v) \geqslant 3$, or connected when $d_G(v) = 2$.*

A spanning subgraph $H$ of $G$ is called a *maximum planar subgraph* of $G$ if $H$ is planar and contains as many edges as possible.