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The two-way rewriting in action: Removing the mystery of Euler–Glaisher's map

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Abstract

Starting with Euler's bijection between the partitions into odd parts and the partitions into distinct parts, one basic activity in combinatorics is to establish partition identities by so-called 'bijective proofs,' which amounts to constructing explicit bijections between two classes of the partitions under consideration.

The aim of this paper is to give a global view on the Glaisher-type bijections and related rewriting maps.

Glaisher's map is a bijection between partitions with no part divisible by m and partitions with no parts repeated m or more times, that uses basic number theoretic techniques. O'Hara's rewriting map is also a bijection between those two sets (the map consists of repeated replacing any m occurrences of a part, say z, by the number mz). It is remarkable that both of these bijections are identical. Moreover, the bijections produced for many partition identities by the refine machinery developed by, for example, Remmel, Gordon, O'Hara, and Sellers, Sills, and Mullen, turn out to be the same bijections as the ones found by Euler and generalized by Glaisher.

Here we give a quite paradoxical answer to the question of why Euler–Glaisher's bijections arise so persistently from their applications, namely: *Whatever Euler-like partition identities we take, one and the same Euler–Glaisher's map will be suited for all of them.*

We prove this by giving an alternate description of the bijections using two-way rewriting bijections between any two equinumerous partition ideals of order 1, provided, as a partial case, by a general criterion from Kanovich [Finding direct partition bijections by two-directional rewriting techniques, Discrete Math. 285(1–3) (2004) 151–166]. The tricky part of the proof is that, generally speaking, Euler–Glaisher's mapping *differs* from the rewriting mapping derived, but both mappings are proved to behave *identically* on all partitions that might have been involved as elements of some equinumerous 'Euler pairs'.

We generalize Glaisher's mapping by simply substituting *mixed radix* expansions for the base m expansions in Glaisher's original construction. With this direct generalization we extend the Euler–Glaisher's phenomenon to any two equinumerous partition ideals of order 1, whenever one of the ideals consists of partitions into parts from a set. As a useful part of the proof, we develop a natural generalization of Andrews–Subbarao's criterion [G.E. Andrews, Two theorems of Euler and a general partition theorem, Proc. Amer. Math. Soc. 20(2) (1969) 499–502; M.V. Subbarao, Partition theorems for Euler pairs, Proc. Amer. Math. Soc. 28(2) (1971) 330–336].

Keywords: Integer partitions; Partition identities; Euler–Glaisher's map; Multiset rewriting; Confluence; Termination; Strong normalization

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1. Motivating examples and summary

An *integer partition* of $n = m_1 + m_2 + \cdots + m_k$ can be identified as a *multiset* M consisting of positive integers $m_1, m_2, \ldots, m_k$. We will represent this M as $M = \{m_1, m_2, \ldots, m_k\}$, where the number of copies of some m shows the multiplicity of the m within M. Each m_i is called a *part* of the partition. This sum $m_1 + m_2 + \cdots + m_k$ will be also denoted by $\|M\|$.

Definition 1.1. Two classes of partitions $\mathscr{C}_1$ and $\mathscr{C}_2$ are called *equinumerous* if $p(\mathscr{C}_1, n) = p(\mathscr{C}_2, n)$, for all n. Here $p(\mathscr{C}, n)$ stands for the number of partitions of n that belong to a given class $\mathscr{C}$.

Starting with Euler's bijection between the partitions into odd parts and the partitions into distinct parts, one basic activity in combinatorics is to establish partition identities by so-called 'bijective proofs,' which amounts to constructing explicit bijections between two classes of the partitions under consideration. A unified method for dealing with a large class of integer partition identities has been developed by Andrews, Garsia and Milne, Remmel, Gordon, Wilf, O'Hara, and others (see [2,18,10]). It is remarkable that the bijections produced for many partition identities by their refine machinery are the same bijections as the ones found by Euler and generalized by Glaisher [6] in pure number theoretic terms. This paper is inspired by the recent results of Sellers, Sills, and Mullen on Glaisher-type bijections [15], and based on techniques developed in Kanovich [10].

The aim of the paper is to show that a novel *two-directional* rewriting technique removes the mystery of certain known results and can establish new results in the theory of integer partitions. In particular, we prove that the appearance of Euler–Glaisher's mapping in many contexts is not accidental and that Euler–Glaisher's mapping is good for *all* Euler-type identities. In particular (see Corollary 5.1):

> *The same Euler–Glaisher's original map always provides a bijection h between equinumerous partition ideals* $\mathscr{C}_1$ *and* $\mathscr{C}_2$,[1] *whenever* $\mathscr{C}_1$ *consists of all partitions into parts taken from some set* S_1, *and* $\mathscr{C}_2$ *consists of all partitions into parts from some set* S_2 *in which each part may occur at most* $m - 1$ *times* (*m is fixed*).

To illustrate the basic features of our approach, consider Euler's partition theorem and its numerous 'relatives':

Example 1.1. For any positive integer n,

(a) Euler: The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.
(b) Glaisher [6]: The number of partitions of n with no part divisible by m equals the number of partitions of n with no part repeated m or more times.
(c) Guy [8]: The number of partitions of n into odd parts greater than 1 equals the number of partitions of n into distinct parts which are not powers of 2.
(d) Schur [14]: The number of partitions of n into parts congruent to $\pm 1(\mathrm{mod})6$ equals the number of partitions of n into distinct parts congruent to $\pm 1(\mathrm{mod})3$.
(e) "1–2": The number of partitions of n into ones equals the number of partitions of n into distinct powers of 2 (both numbers are equal to 1).
(f) '2·Euler': The number of partitions of n into parts congruent to $2 \bmod 4$ equals the number of partitions of n into distinct even parts.

It should be noted that Example 1.1 is a simple corollary of Andrews' criterion for partition ideals of order 1 [2, Theorem 8.4], which was proved via generating functions.

However, to reveal the essence and get a broader understanding of the partition identities, bijective proofs are preferable. As for item (a), Euler himself established an explicit bijection between the partitions into odd parts and the partitions into distinct parts in pure number theoretic terms (see Comment 1.2): the basic ingredient of Euler's map—that every number has a unique binary representation, is presented in item (e). Euler's map was generalized by Glaisher [6] to cover item (b) in Example 1.1. Recently Sellers, Sills, and Mullen have shown Glaisher-type bijections for item (c) and its generalizations [15].

[1] We say "*provides a bijection h between two classes of integer partitions* $\mathscr{C}_1$ *and* $\mathscr{C}_2$" meaning that, for any n, function h is a bijection between the partitions of n that belong to $\mathscr{C}_1$ and the partitions of n that belong to $\mathscr{C}_2$.