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Note

Path extendability of claw-free graphs☆

Yu Sheng[a], Feng Tian[a], Jianglu Wang[b], Bing Wei[a,c,1], Yongjin Zhu[a]

[a] *Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China*
[b] *Department of Mathematics, Shangdong Normal University, Jinan 250014, China*
[c] *Department of Mathematics, University of Mississippi, MS 38677, USA*

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Abstract

Let G be a connected, locally connected, claw-free graph of order n and x, y be two vertices of G. In this paper, we prove that if for any 2-cut S of G, $S \cap \{x, y\} = \emptyset$, then each (x, y)-path of length less than $n-1$ in G is extendable, that is, for any path P joining x and y of length $h(<n-1)$, there exists a path P' in G joining x and y such that $V(P) \subset V(P')$ and $|P'| = h+1$. This generalizes several related results known before.

Keywords: Claw-free graphs; Detour; Locally connected; Path extendable

1. Introduction and main results

We consider only finite, simple and connected graphs. For terminology and notation not defined here we refer to [2]. Throughout this paper, let G be a graph of order n, $V(G)$ and $E(G)$ denote, respectively, the vertex set and the edge set of G. For each vertex u of G, the neighborhood $N(u)$ of u is the set of all vertices adjacent to u. Set $N[u] = N(u) \cup \{u\}$. For $S \subseteq V(G)$, denote by $G[S]$ the subgraph of G induced by S. For convenience, let $H_u = G[N(u)]$. A vertex u of G is said to be *locally connected* if H_u is connected. G is called *locally connected* if each vertex of G is locally connected. Generally, G is called *locally k-connected* if for each vertex u, H_u is k-connected. A connected, locally k-connected graph must be $(k+1)$-connected. The distance between two vertices x, y is denoted by $d(x, y)$. A k-cut is a cut set containing k vertices.

A path with end vertices x and y is called an (x, y)-path. An (x, y)-path P is *extendable* if there is an (x, y)-path P' in G such that $V(P') \supset V(P)$ and $|V(P')| = |V(P)| + 1$. In this case we say also that P can be extended to P'. An (x, y)-path is a hamiltonian path of G if it contains all the vertices of G. A graph G is said to be *path extendable* if for each pair of vertices x, y and for each nonhamiltonian (x, y)-path P in G, P is extendable.

A graph G is said to be *hamiltonian* if it has a cycle containing all the vertices of G. G is *panconnected* if each pair of distinct vertices x, y are joined by a path of length h for each h, $d(x, y) \leqslant h \leqslant n-1$. A graph G is called claw-free if it has no induced subgraph isomorphic to $K_{1,3}$.

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Many results on hamiltonian properties of claw-free graphs have appeared during the last two decades. We refer the reader to a recent survey [5]. In this paper, we are interested in some results involving the local connectivity of a claw-free graph. In 1979, Oberly and Sumner [7] proved that every connected, locally connected, claw-free graph G of order $n \geqslant 3$ is hamiltonian. Clark [4] improved this result by showing that in a graph G satisfying the same condition, each vertex of G lies on a cycle of length from 3 to n inclusive. Under the condition that G is locally 2-connected, Kanetkar and Rao [6] and Wang and Zhu [9] got stronger properties of G, respectively.

Theorem 1 (*Kanetkar and Rao [6]*). *Every connected, locally 2-connected, claw-free graph is panconnected.*

Theorem 2 (*Wang and Zhu [9]*). *Every connected, locally 2-connected, claw-free graph is path extendable.*

The question is whether the locally 2-connectedness can be replaced by locally connectedness without changing those properties of G. In [8], Sheng et al. proved the following result:

Theorem 3 (*Sheng et al. [8]*). *Let G be a connected, locally connected, claw-free graph of order n and x, y be any two vertices of G. If for any 2-cut S, $S \cap \{x, y\} = \emptyset$, then x and y are joined by a path of length h for each h, $d(x, y) \leqslant h \leqslant n-1$.*

Theorem 3 generalized Theorem 1 and solved the following conjecture proposed by Broersma and Veldman:

Conjecture 4 (*Broersma and Veldman [3]*). Let G be a connected, locally connected, claw-free graph of order at least 4. Then G is panconnected if and only if G is 3-connected.

In this paper, we show the following, which generalizes all results mentioned above:

Theorem 5. *Let G be a connected, locally connected, claw-free graph of order n and x, y be any two vertices of G. If for any 2-cut S, $S \cap \{x, y\} = \emptyset$, then each (x, y)-path of length less than $n-1$ of G is extendable.*

It is easy to see that if for given two vertices x and y of a graph G, G contains (x, y)-paths of all possible lengths, then $\{x, y\}$ must not be a 2-cut. We will construct a connected, locally connected, claw-free graph G_0 to show that the condition "for any 2-cut S, $S \cap \{x, y\} = \emptyset$" cannot be replaced either by "$\{x, y\}$ is not a 2-cut" or by "$S \cap \{x\} = \emptyset$". G_0 is a graph consisting of three distinct complete graphs G_1, G_2 and G_3 with $|G_i| \geqslant 3$ for $1 \leqslant i \leqslant 3$, $|G_i \cap G_j| = 1$ for $1 \leqslant i < j \leqslant 3$ and $G_1 \cap G_2 \cap G_3 = \emptyset$. Obviously, G_0 is a connected, locally connected, claw-free graph. Taking $x \in G_3 \backslash (G_1 \cup G_2)$ and $y \in G_1 \cap G_2$, there is no hamiltonian (x, y)-path in G_0.

2. Several lemmas

Let $P = x_1 x_2 \cdots x_p$ be an (x_1, x_p)-path of G with an orientation from x_1 to x_p. We let $x_i P x_j$, for $1 \leqslant i \leqslant j \leqslant p$, be the subpath $x_i x_{i+1} \cdots x_j$, and $x_j \overline{P} x_i = x_j x_{j-1} \cdots x_i$. We will consider $x_i P x_j$ and $x_j \overline{P} x_i$ both as paths and as vertex sets. We put $x_i^{-(P)} = x_{i-1}$, $x_i^{+(P)} = x_{i+1}$, $x_i^{-2(P)} = x_{i-2}$ and $x_i^{+2(P)} = x_{i+2}$. If there is no doubt about the path we only write x_i^-, x_i^+, etc. We say that an (x, y)-path P is minimal if there is no (x, y)-path P' in G such that $V(P') \subset V(P)$.

Let z be an internal vertex of an (x, y)-path P. If there exists a minimal (u_1, u_s)-path $Q = u_1 u_2 \cdots u_s$ in H_z such that $u_2 = x$, $u_s = z^+$ ($u_2 = y$, $u_s = z^-$, resp.) and u_1 is the only vertex of Q not contained in P, then we call z an x-detour (a y-detour, resp.) vertex of P, and Q a (z, x)-detour (a (z, y)-detour, resp.) of P. In this case, we say also that P has a (z, x)-detour (a (z, y)-detour, resp.). By the definition, if Q is a (z, x)-detour or a (z, y)-detour of P then the order of Q is at least 3.

We emphasis that for any $z \in V(G)$, the order of any minimal path in H_z is at most 4, if G is a claw-free graph. We will use the following lemmas which were proved in [8].

Lemma 1 (*Sheng et al. [8]*). *Let G be a claw-free graph, P be an (x, y)-path of G and z be an internal vertex of P with $N(z) \nsubseteq V(P)$. If P is not extendable and z is a locally connected vertex, then P has a (z, x)-detour or a (z, y)-detour.*