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Decompositions of signed-graphic matroids

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Abstract

We give a decomposition theorem for signed graphs whose frame matroids are binary and a decomposition theorem for signed graphs whose frame matroids are quaternary.

Keywords: Martroid; Signed graph

1. Introduction

Throughout this paper we will assume that the reader is familiar with matroid theory as in [6]. The reader may or may not be familiar with signed graphs as in [17]. If not, we give an overview of all necessary information for signed graphs in Section 2.

Signed graphs and signed-graphic matroids have received and continue to receive much attention in the mathematical literature. (See, for example, [1,3,5,8,9,11,12,20].) Signed-graphic matroids have the potential to be a well-understood class of matroids much like the class of graphic matroids. It is even conjectured (in [16, Section 4]) that signed-graphic matroids may decompose the classes of near-regular matroids and dyadic matroids in much the same way that graphic matroids decompose the class of regular matroids in Seymour's Decomposition Theorem (see [10]). Thus more knowledge of the structure of signed-graphic matroids is desirable. One very basic matter is to understand their representability properties over various fields.

A signed graph is a pair $\Sigma = (G, \sigma)$ where G is a graph and σ is a function from the edges of G to the multiplicative group $\{+1, -1\}$. A circle (i.e., a simple closed path) in Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle is called *negative*. The frame matroid of Σ (first studied by Zaslavsky in [17]) is the matroid on the edges of Σ whose circuits are edge sets of positive circles and edge sets of subgraphs that are subdivisions of the graphs shown in Fig. 1 and contain no positive circles. We will call such a matroid a *signed-graphic matroid*. Signed-graphic matroids are precisely the Dowling geometries and their minors for the group of order two.

Theorem 1.1 is from [17, Theorem 8B.1]. (See also Theorem 1.1 in Section 2.)

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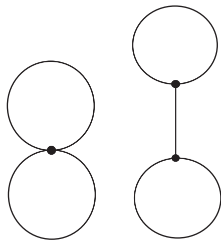

Fig. 1.

Theorem 1.1 (*Zaslavsky*). *The matroid $M(\Sigma)$ is representable over any field of characteristic not equal to* 2.

So it only remains to determine when $M(\Sigma)$ is representable over fields of characteristic two. It is shown in [15] that if M is representable over $GF(3)$, $\mathbb{Q}$, and a field of characteristic two, then M is representable over all fields except maybe $GF(2)$. So it only remains to find when $M(\Sigma)$ is binary (i.e., representable over $GF(2)$) and when $M(\Sigma)$ is quaternary (i.e., representable over $GF(4)$).[2]

Our main results are those shown in Sections 1.1 and 1.2. Most all of the work for proving these results is done in Gerards' monograph [5, Section 3.2] and Pagano's doctoral dissertation [8, Chapter 2]. In this paper, we survey and connect the pertinent information in [5,8] and prove some other lemmas in order to form the results of Sections 1.1 and 1.2.

The rest of this paper is organized as follows. In Section 2 we have our definitions. In Section 3 we define and discuss a notion of k-sums of signed graphs, their connection to matroid k-sums, and some applications. In Section 4 we give the proofs of our main results.

1.1. Binarity

A signed graph is called *balanced* when it has no negative circles. A *balancing vertex* in an unbalanced signed graph Σ is a vertex whose removal leaves a balanced subgraph. A signed graph is *joint unbalanced* if it is balanced after the removal of all negative loops. Negative loops are called *joints*, which is a term taken from the theory of Dowling geometries. A unbalanced signed graph is called *tangled* if it has no balancing vertex and no two vertex-disjoint negative circles.

Theorem 1.2. *If $M(\Sigma)$ and $M(\Upsilon)$ are both binary, then for each $k \in \{1, 2, 3\}$, $M(\Sigma \oplus_k \Upsilon) = M(\Sigma) \oplus_k M(\Upsilon)$ is binary.*

Theorem 1.3. *If Σ is connected and $M(\Sigma)$ is binary, then either*

(1) *Σ is balanced,*
(2) *Σ is joint unbalanced,*
(3) *Σ has a balancing vertex,*
(4) *Σ is tangled, or*
(5) *$\Sigma = \Upsilon_1 \oplus_k \Upsilon_2$ for some $k \in \{1, 2\}$ where each $M(\Upsilon_i)$ is binary.*

Also, if Σ is a connected signed graph that satisfies one of (1)–(4), *then $M(\Sigma)$ is binary.*

Later (in Propositions 2.2 and 2.3) we show that signed graphs from parts (1)–(3) in Theorem 1.3 have matroids that are graphic via some canonical transformations. Since the class of graphic matroids is closed under k-summing, signed graphs obtained by k-sums of the types in parts (1)–(3) have graphic matroids. So the question of when $M(\Sigma)$

[2] Given Theorem 1.1 and the discussion in the introduction of [15], $M(\Sigma)$ is binary iff $M(\Sigma)$ is regular and $M(\Sigma)$ is quaternary iff $M(\Sigma)$ is near regular.