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Note

Chordal multipartite graphs and chordal colorings

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Abstract

'Chordal multipartite graphs' are properly colored graphs such that two vertices in a minimal vertex separator are adjacent if and only if they are differently colored. They have induced cycle characterizations that transcend those of chordal and chordal bipartite graphs. Graphs that have such 'chordal colorings' are weakly chordal graphs with simple forbidden subgraph characterizations, and such a chordal coloring of a graph G requires only $\chi(G)$ colors.

Keywords: Chordal graph; Vertex separator; Chordal bipartite graph; Coloring

1. Introduction

Recall that a *proper coloring* of a graph G is a vertex coloring in which adjacent vertices get different colors, and an *optimal coloring* of G is a proper coloring that uses the minimum possible number, $\chi(G)$, of colors. A set $S \subset V(G)$ is a u, v*-separator* if vertices u and v are in the same component of G, but different components of $G - S$; if S is also inclusion-minimal, then S is a *minimal* u, v*-separator*. More generally, S is a (*minimal*) *vertex separator* if there exist vertices u and v such that S is a (*minimal*) u, v*-vertex separator*.

Define a properly k-colored graph G, $\chi(G) \leqslant k \leqslant |V(G)|$, to be a *chordal k-partite graph* if every minimal vertex separator $S \subset V(G)$ induces a subgraph in which vertices are adjacent if and only if they are differently colored (but not all of the k colors need to occur in S); G is a *chordal multipartite graph* if there exists a k for which G is a chordal k-partite graph. The example in Fig. 1 shows that different proper k-colorings of the same graph can be chordal multipartite or not (this graph has three minimal vertex separators, each consisting of a pair of vertices in the 4-cycle, at least one having degree 3).

The chordal 2-partite graphs as just defined are precisely the traditional [1,2,4,7] *chordal bipartite graphs*—the bipartite graphs in which every induced cycle is a 4-cycle; equivalently, these are the bipartite graphs in which every cycle large enough to have a chord does have a chord. (These are just called 'chordal graphs' in [1], which is entirely within the context of bipartite graphs.) The classes of chordal 2-partite graphs and chordal bipartite graphs are identical because of the Golumbic–Goss characterization of chordal bipartite graphs as the graphs in which every minimal vertex separator induces a complete bipartite graph (possibly an independent set, all with the same color). (Note: The Golumbic–Goss characterization is usually stated in terms of minimal edge separators, but it also holds for minimal vertex separators, as noted in [8].)

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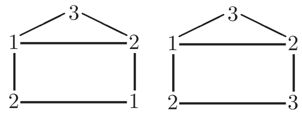

Fig. 1. Two properly 3-colored graphs; the one on the left is chordal multipartite, but not the one on the right—the color-1 vertex and the lower-right color-3 vertex induce a minimal vertex separator in which differently colored vertices are not adjacent.

The chordal $|V(G)|$-partite graphs as defined above are precisely the traditional [2,4,7] *chordal graphs*—the graphs in which every induced cycle is a 3-cycle; equivalently, every cycle large enough to have a chord does have a chord. The classes of chordal $|V(G)|$-partite graphs and chordal graphs are identical because of the Dirac characterization of chordal graphs as the graphs in which every minimal vertex separator induces a complete graph. Chordal graphs are also chordal k-partite whenever $\chi(G) \leqslant k \leqslant |V(G)|$. (Warning: Despite their names, chordal k-partite graphs, chordal bipartite graphs, and chordal multipartite graphs are almost never chordal graphs; the end of the 'Epilogue 2004' chapter of [4] defends the terminology.)

Section 2 will present several basic results concerning chordal multipartite graphs, beginning with an induced cycle characterization in Theorem 1 that extends the traditional induced cycle definitions of chordal bipartite graphs and chordal graphs. Theorem 4 and Corollary 5 will characterize the graphs that are chordal k-partite for some proper k-coloring, and Theorem 6 will show that only $k = \chi(G)$ colors are needed.

2. Main results

Theorem 1. *A properly k-colored graph is chordal k-partite if and only if every induced cycle is either a* 3*-cycle or a* 2*-colored* 4*-cycle.*

Proof. First, suppose a properly k-colored graph G is chordal k-partite, and $C = v_1, v_2, \ldots, v_l, v_1$ is an induced l-cycle of G. The $l = 3$ case is immediate. Suppose either $l = 4$ and C is not 2-colored or that $k \geqslant 5$ (arguing toward a contradiction). Choose v_i and v_j to be nonadjacent and differently colored. Then v_i and v_j are in a common minimal $v_{i-1}v_{i+1}$-separator (setting $v_{l+1} = v_1$), contradicting the definition of chordal k-partite.

Conversely, suppose G is properly k-colored but not chordal k-partite; say S is a minimal x, y-separator that contains differently colored yet nonadjacent vertices u and v, and G_x and G_y are the components of $G - S$ that contain, respectively, x and y. By the minimality of S, there exist both an x, y-path that contains u but not v and a second x, y-path that contains v but not u. The portions of these two paths that lie in G_x determine an induced u, v-path π whose interior vertices all lie in G_x; similarly, there is an induced u, v-path π' whose interior vertices all lie in G_y. Then, $\pi \cup \pi'$ will form an induced cycle C in G that passes through u and v and has $|C| \geqslant 4$. Because u and v have different colors, $|C| = 4$ would require C to contain at least three different colors of vertices. Thus, C is neither a 3-cycle nor a 2-colored 4-cycle. □

A *long hole* is an induced cycle of length greater than four. Theorem 1 implies that chordal multipartite graphs have no long holes. Theorem 2 below will be comparable to the characterization [6] of graphs that have no long holes as the graphs in which every *l-cycle* (meaning every cycle, not necessarily induced, of length $l \geqslant 3$) is the sum of c_3 distinct 3-cycles and c_4 distinct 4-cycles such that $c_3 + 2c_4 = l - 2$ (where the *sum of cycles* means the symmetric difference of their edge sets, as is traditional for cycle spaces of graphs).

Theorem 2. *A properly k-colored graph is chordal k-partite if and only if every l-cycle*, $l \geqslant 3$, *is the sum of* c_3 *distinct* 3*-cycles and* c_4^* *distinct* 2*-colored* 4*-cycles such that* $c_3 + 2c_4^* = l - 2$.

Proof. First, suppose G is a properly k-colored, chordal k-partite graph, and C is an l-cycle, $l \geqslant 3$. If $l = 3$, then simply set $c_3 = 1$ and $c_4^* = 0$; if $l = 4$, then C is 2-colored by Theorem 1 so simply set $c_3 = 0$ and $c_4^* = 1$. So suppose $l \geqslant 5$. By Theorem 1, C must have a chord e such that C is the sum of cycles C_1 and C_2 of lengths l_1 and l_2 respectively, where $\{e\} = E(C_1) \cap E(C_2)$ and $l_1 + l_2 = l + 2$. Induction implies that each C_i is the sum of c_{i3} distinct 3-cycles and c_{i4}^* distinct 2-colored 4-cycles such that each $c_{i3} + 2c_{i4}^* = l_i - 2$. Hence, C is the sum of $c_3 = c_{13} + c_{23}$ distinct 3-cycles and $c_4^* = c_{14}^* + c_{24}^*$ distinct 2-colored 4-cycles such that $c_3 + 2c_4^* = (l_1 - 2) + (l_2 - 2) = l - 2$.