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Note

Blocking sets in line Grassmannians

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Abstract

In this paper the most natural questions concerning the blocking sets in the line Grassmannian of $\mathrm{PG}(n,q)$ are partially answered. In particular, the following Bose–Burton type theorems are proved: if n is odd or $n=4$, then the blocking sets of minimum size are precisely the linear complexes with singular subspace of minimum dimension.

Keywords: Blocking set; Grassmann variety; Linear complex

1. Introduction

In 1980, Tallini [10] developed the theory of k-sets in the line Grassmannian of a Galois space $\mathrm{PG}(n,q)$, as a natural extension of the analogous one in $\mathrm{PG}(n,q)$. Since 1999, Metsch [5,6] proved several results characterizing some hyperplane sections of polar spaces as being the smallest blocking sets. He called such results "Bose–Burton type theorems" in analogy with the well-known characterization of the linear subspaces of $\mathrm{PG}(n,q)$ [1]. The aim of this paper is to develop a theory of blocking sets in the line Grassmannian of $\mathrm{PG}(n,q)$ taking inspiration from the mentioned results.

In this note we deal with the projective space $\mathrm{PG}(n,q)$ coordinatized by $\mathbb{F}_q$, where $n>2$, and q is a prime power. Let $\mathscr{P}$ and $\mathscr{R}$ denote the point and line set of $\mathrm{PG}(n,q)$, respectively. If $A\in\mathscr{P}$ and π is a plane of $\mathrm{PG}(n,q)$ containing A, the *pencil* of lines with *center* A and *support* π is the set $\varphi(A,\pi)$ of all lines through A, lying on π. Let $\mathscr{F}$ denote the set of all pencils. The geometry $\Gamma(n,1,q)=(\mathscr{R},\mathscr{F})$ is a semilinear space, known as the *line Grassmannian* of $\mathrm{PG}(n,q)$. Occasionally, we will call the elements of $\mathscr{R}$ *G-points*, and the elements of $\mathscr{F}$ *G-lines*. Analogously, planes and projective subspaces contained in $(\mathscr{R},\mathscr{F})$ will be called *G-planes* and *G-subspaces*, respectively.

More generally, a *pencil of h-subspaces* in $\mathrm{PG}(n,F)$, F a field, is the set of all h-subspaces containing a given $(h-1)$-subspace, say L, and contained in a given $(h+1)$-subspace M, with $L\subseteq M$. The hth *Grassmannian* $\Gamma(n,h,F)$ of $\mathrm{PG}(n,F)$, $0\leqslant h\leqslant n-1$, is the semilinear space whose points are the h-subspaces in $\mathrm{PG}(n,F)$ and whose lines are the pencils of h-subspaces.

Let $A,B\in\mathscr{P}$ be two distinct points and $\ell=AB$ the line joining them. Let us represent such points by coordinates, say $A=\mathbb{F}_q(x_0,x_1,\dots,x_n)$ and $B=\mathbb{F}_q(y_0,y_1,\dots,y_n)$. The elements

$$p_{ij}=x_iy_j-x_jy_i,\quad 0\leqslant i<j\leqslant n,$$

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define a point $\mathbb{F}_q(p_{ij})$ of $\mathrm{PG}(N, q)$, where

$$N = \binom{n+1}{2} - 1. \tag{1}$$

This point is independent on the choice of $A, B \in \ell$, $A \neq B$, and their coordinates, so it can be uniquely associated with ℓ. The map $\wp : \ell \mapsto \mathbb{F}_q(p_{ij})$ is the well-known *Plücker embedding*, and the elements p_{ij} are the so-called *Plücker coordinates* of ℓ. The set $\mathscr{G}_{n,1,q} = \mathscr{R}^{\wp}$ is the *Grassmann variety* representing the lines of $\mathrm{PG}(n, q)$. For $n = 3$, this variety is also known as the *Klein quadric*. The Plücker embedding is one-to-one, furthermore the image of an element of $\mathscr{F}$ is a line of $\mathrm{PG}(N, q)$ contained in $\mathscr{G}_{n,1,q}$, and conversely. By these properties of $\wp$, the semilinear space whose point set is $\mathscr{G}_{n,1,q}$ and whose lines are the lines of $\mathrm{PG}(N, q)$ contained in $\mathscr{G}_{n,1,q}$ is isomorphic to $\Gamma(n, 1, q)$.

Let $A \in \mathscr{P}$, and let S_A be the *star of lines* with center A, i.e.

$$S_A = \{\ell \in \mathscr{R} | A \in \ell\}.$$

This S_A is an $(n-1)$-dimensional G-subspace, and $S_A^{\wp}$ is a subspace of $\mathrm{PG}(N, q)$ entirely contained in $\mathscr{G}_{n,1,q}$. We define $\mathscr{S} = \{S_A | A \in \mathscr{P}\}$. Next, let π be a plane of $\mathrm{PG}(n, q)$, and $T_\pi = \{\ell \in \mathscr{R} | \ell \subset \pi\}$. Such T_π, which will be called a *ruled plane*, is a G-plane, and $T_\pi^{\wp}$ is a plane of $\mathrm{PG}(N, q)$ contained in $\mathscr{G}_{n,1,q}$. Let $\mathscr{T}$ be the set of all ruled planes. It is quite clear that $\mathscr{S} \cup \mathscr{T}$ is the set of all maximal G-subspaces of $\Gamma(n, 1, q)$. So, every subspace of $\mathrm{PG}(N, q)$ contained in $\mathscr{G}_{n,1,q}$, maximal with respect to inclusion, is the image under $\wp$ of an element of $\mathscr{S} \cup \mathscr{T}$, and conversely.

A *linear complex* in $\mathrm{PG}(n, q)$ is the set of all lines whose Plücker coordinates satisfy a non-vanishing homogeneous linear equation

$$\sum_{\substack{i,j=0 \\ i<j}}^{n} \alpha_{ij} p_{ij} = 0. \tag{2}$$

Let $\mathbf{B}$ be a linear complex. If a point P exists such that all lines through P belong to $\mathbf{B}$, then the linear complex is called *degenerate*; otherwise, $\mathbf{B}$ is *general*.

A *null polarity* of $\mathrm{PG}(n, q)$ is a mapping ω, which maps a point A, represented by a column vector $\mathbf{a}$, onto a hyperplane A^{ω} of coordinates $\mathbf{u} = M\mathbf{a}$, where $M \neq O$ is a given skew-symmetric $(n+1) \times (n+1)$ matrix. The mapping is defined only on the *non-singular points*, that is when $M\mathbf{a} \neq O$. If M is singular, then ω is said to be *degenerate*. There is a non-degenerate null polarity in $\mathrm{PG}(n, q)$ if and only if n is odd. The following result is well known (cf. e.g. [8, Section 156]):

Result 1. *Let* $\mathbf{B}$ *be the linear complex of Eq.* (2), *and denote by* ω *the null polarity of* $\mathrm{PG}(n, q)$ *with matrix* $M = (\alpha_{ij})$, *where* $\alpha_{ij} = 0$ *for* $i = j$ *and* $\alpha_{ij} = -\alpha_{ji}$ *for* $i > j$. *Let A and B be two distinct points. Then the line AB belongs to* $\mathbf{B}$ *if and only if either A is a singular point for* ω, *or* $B \in A^{\omega}$. *This defines a bijection between the set of all linear complexes and the set of all null polarities, and degenerate linear complexes are related to degenerate null polarities.*

A *blocking set* in a semilinear space $(\mathscr{R}, \mathscr{F})$ is a set $\mathbf{B} \subseteq \mathscr{R}$, such that every $\varphi \in \mathscr{F}$ meets $\mathbf{B}$. Our goal is to study the blocking sets of $\Gamma(n, 1, q)$. In the following sections some problems concerning blocking sets in $\Gamma(n, 1, q)$ are partially answered. This can be seen as the first step in investigating the blocking sets with respect to the k-subspaces in $\Gamma(n, h, q)$, where $0 < h < n - 1$, $k \leqslant \max\{n - h, h + 1\}$, which are currently object of investigation.

We shall use the notation

$$\theta_i = \frac{q^{i+1} - 1}{q - 1}, \quad i \in \mathbb{N} \cup \{-1\}.$$

2. Examples of blocking sets in $\Gamma(n, 1, q)$

If $\mathbf{B}'$ is a blocking set with respect to lines in $\mathrm{PG}(N, q)$, then $\mathbf{B} = (\mathbf{B}' \cap \mathscr{G}_{n,1,q})^{\wp^{-1}}$ is a blocking set in $\Gamma(n, 1, q)$. In case $\mathbf{B}'$ is a hyperplane of $\mathrm{PG}(N, q)$, then $\mathbf{B}$ is a linear complex. Such a complex is degenerate if and only if $\mathbf{B}$ contains some $S \in \mathscr{S}$.