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Bi-cyclic decompositions of complete graphs into spanning trees☆

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Abstract

We examine decompositions of complete graphs K_{4k+2} into $2k+1$ isomorphic spanning trees. We develop a method of factorization based on a new type of vertex labelling, namely blended ρ-labelling. We also show that for every $k \geqslant 1$ and every d, $3 \leqslant d \leqslant 4k+1$ there is a tree with diameter d that decomposes K_{4k+2} into $2k+1$ factors isomorphic to T.

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1. Introduction

Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. The decomposition is *cyclic* if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of vertices of K_n and isomorphisms $\phi_i : G_0 \to G_i$, $i = 1, 2, \ldots, s$ such that $\phi_i(x_j) = x_{i+j}$ for every $j = 1, 2, \ldots, n$, where the subscripts are taken modulo n. If G has exactly n vertices and none of them is isolated, then G is called *factor* and the decomposition is called *G-factorization of K_n*.

Graph decompositions, most often isomorphic decompositions of complete graphs, have been extensively studied. In particular, decompositions of complete graphs and complete bipartite graphs into isomorphic trees of smaller order were studied by many authors. Surprisingly enough, almost nothing was published on factorizations of complete graphs into isomorphic spanning trees. A simple arithmetic condition shows that only complete graphs with an even number of vertices can be factorized into spanning trees. It is a well-known fact that each such graph K_{2n} can be factorized into hamiltonian paths P_{2n}. On the other hand, it is easy to observe that each K_{2n} can be also factorized into double stars; that is, two stars $K_{1,n-1}$ joined by an edge. But what about trees between these two extremal cases? In [2], Eldergill developed a method of T-factorization of K_{2n} into symmetric trees using two types of graph labellings based on labellings introduced earlier by Rosa [9,10]. Here by a *symmetric* tree we mean a tree with an automorphism ψ and

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an edge xy such that $\psi(x)=y$ and $\psi(y)=x$. In [3] the author defined a *flexible q-labelling* that allows T-factorization of K_{4k+2} into certain classes of trees; in [4] the author with Kubesa generalized this labelling for trees with $4k$ vertices.

We present here another type of labelling, which generalizes properties of a ρ-labelling and a graceful labelling (also called β-labelling) introduced by Rosa [10] and some other labellings (see, e.g., [6,8]). Using this labelling, we then present a recursive procedure that produces infinite families of trees that factorize complete graphs.

2. Blended ρ-labelling and blended graceful labelling

As we mentioned above, Rosa introduced some important types of vertex labellings that we list now. Graceful labelling (also called β-labelling) and ρ-labelling are being used for decompositions of complete graphs K_{2n+1} into graphs with n edges. We define ρ-labelling in a slightly different manner which suits better our further needs. We define a *labelling* of a graph G with n edges as an injection λ from the vertex set of G, $V(G)$, into a subset S of the set $\{0,1,2,\ldots,2n\}$. Later we will use more general definition. The *length* of an edge (x,y) is defined as $\ell(x,y)=\min\{|\lambda(x)-\lambda(y)|, 2n+1-|(\lambda(x)-\lambda(y))|\}$. If the set of all lengths of the n edges is equal to $\{1,2,\ldots,n\}$ and $S\subseteq\{0,1,\ldots,2n\}$, then λ is ρ-*labelling*; if $S\subseteq\{0,1,\ldots,n\}$ instead, then λ is *graceful* or β-*labelling*. A graceful labelling λ is said to be α-*labelling* if there exists a number λ_0 with the property that for every edge $(x,y)\in G$ with $\lambda(x)<\lambda(y)$ it holds that $\lambda(x)\leqslant\lambda_0<\lambda(y)$. For an exhaustive survey of graph labellings, see Gallian [5].

Each graceful labelling is indeed also a ρ-labelling. One can observe that if a graph G with n edges has a graceful labelling or ρ-labelling, then K_{2n+1} can be cyclically decomposed into $2n+1$ copies of G. It is so because K_{2n+1} has exactly $2n+1$ edges of length i for every $i=1,2,\ldots,n$ and each copy of G contains exactly one edge of each length.

Graceful labelling and ρ-labelling can be used to produce a new labelling allowing factorizations of K_{4k+2} into $2k+1$ copies of a tree T with $4k+1$ edges. To simplify our notation, we often unify vertices with their respective labels. We will say "a vertex i" rather than "a vertex x with $\lambda(x)=i$". We will also say that a graph *is* graceful rather than that it has a graceful labelling. Although the labellings below could be defined for general graphs as well, we will restrict our definition to trees as we are only interested in factorizations into spanning trees.

First, we define a bipartite version of the cyclic factorization. Let $K_{n,n}$ be a complete bipartite graph with partite sets X and Y. A G-decomposition of $K_{n,n}$ into $G_0, G_1, \ldots, G_{n-1}$ is *bi-cyclic* if there exists an ordering $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ of vertices of $K_{n,n}$ and isomorphisms $\phi_i: G_0\to G_i$, $i=1,2,\ldots,n-1$ such that $\phi_i(x_j)=x_{i+j}$ and $\phi_i(y_j)=y_{i+j}$ for every $j=1,2,\ldots,n$, where the subscripts are taken modulo n.

In this section we develop another method of labelling which is inspired by methods often used in design theory. The main idea is the following. We take a graph K_{4k+2} and split it into three graphs—two copies of K_{2k+1} and one copy of the complete multipartite graph $K_{2k+1,2k+1}$. Then we cyclically decompose each copy of K_{2k+1} into $2k+1$ isomorphic graphs with k edges and bi-cyclically $K_{2k+1,2k+1}$ into $2k+1$ isomorphic graphs with $2k+1$ edges. We have to be careful about our choice of the respective graphs in order to be able to "glue" them together to form a tree. The methods of decomposition of both K_{2k+1} and $K_{2k+1,2k+1}$ are again based on known vertex labellings. We now relax the definition of labelling by allowing labels from the set $\{0_0, 1_0, \ldots, (2k)_0, 0_1, 1_1, \ldots, (2k)_1\}$ rather than from $\{0,1,\ldots,4k+1\}$.

Definition 1. Let G be a graph with $4k+1$ edges, $V(G)=V_0\cup V_1$, $V_0\cap V_1=\emptyset$, and $|V_0|=|V_1|=2k+1$. Let λ be an injection, $\lambda: V_i\to\{0_i, 1_i, 2_i, \ldots, (2k)_i\}$, $i=0,1$. We define the *pure length* of an edge (x_i,y_i) with $x_i, y_i\in V_i$, $i\in\{0,1\}$ as $\ell_{ii}(x_i,y_i)=\min\{|\lambda(x_i)-\lambda(y_i)|, 2k+1-|(\lambda(x_i)-\lambda(y_i))|\}$ for $i=0,1$ and the *mixed length* of an edge (x_0,y_1) as $\ell_{01}(x_0,y_1)=(\lambda(y_1)-\lambda(x_0)) \bmod 2k+1$ for $x_0\in V_0$, $y_1\in V_1$. We say that G has a *blended ρ-labelling* if

(i) $\{\ell_{ii}(x_i,y_i) | (x_i,y_i)\in E(G)\}=\{1,2,\ldots,k\}$ for $i=0,1$,
(ii) $\{\ell_{01}(x_0,y_1) | (x_0,y_1)\in E(G)\}=\{0,1,2,\ldots,2k\}$.

If all mixed edges (x_0,y_1) have $x,y\in\{0,1,\ldots,k\}$ and all pure edges (x_0,y_0) and (x_1,y_1) have $x,y\in\{k,k+1,\ldots,2k\}$ then the labelling is a *blended graceful labelling*. It is a simple observation that a tree with blended graceful labelling consists of three trees. Both graphs induced by the pure edges (x_0,y_0) and (x_1,y_1) are graceful trees themselves. The tree induced by the mixed edges (x_0,y_1) has actually labelling that is equivalent to α-labelling. One can