Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 307 (2007) 78 – 87
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Chromatic sums of general maps on the sphere and the projective plane ☆

Zhaoxiang Li [a], Yanpei Liu [b]

[a] *Department of Mathematics, Central University for Nationalities, Beijing 100081, PR China*
[b] *Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China*

Received 20 July 2004; received in revised form 28 February 2006; accepted 12 May 2006
Available online 25 September 2006

Abstract

In this paper, we study the chromatic sum functions of rooted general maps on the sphere and the projective plane. The chromatic sum function equations of such maps are obtained. From the chromatic sum equations of such maps, the enumerating function equations of rooted loopless maps, bipartite maps and Eulerian maps are also derived. Moreover, some explicit expressions of enumerating functions are also derived.

MSC: 05C10; 05C30; 05C45

Keywords: Chromatic sum function; General maps; Enumerating function

1. Introduction

The chromatic polynomials has been studied by Loerinc [8], Read and Whitehead [9]. On chromatic sums, the first paper which was published in 1973 by Tutte [10] is for rooted planar triangulations. Since 1973, Tutte has published a series of paper [11–14] on chromatic sums for rooted planar triangulations to tackle the coloring average problem. About 10 years later, Liu [5,4] has studied the chromatic sum of rooted nonseparable maps on the plane. All results of chromatic sums having been published are on the plane. Chromatic sums of nonplanar maps are more difficult to be determined. And they are more general and more difficult than enumeration, because of the occurrence of chromatic polynomials. Chromatic sums play an important role in the study of some coloring average. In this paper, we study the chromatic sum of rooted general maps on the sphere and the projective plane.

Let C be a circuit (or curve) on a surface Σ. If $\Sigma - C$ has a connected region homeomorphic to a disc, then C is called *trivial* (or *contractible* as some scholars defined it); otherwise, it is *essential* (or *noncontractible*).

☆ Supported by the NNSFC (60373030), and supported by the project on the tenth five years plan for Central University for Nationalities.
E-mail address: zhaoxiangli8@163.com (Z. Li).

0012-365X/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.disc.2006.05.039

Let $\mathscr{M}$ and $\mathscr{P}$ be, respectively, the set of all rooted general maps on the sphere and the projective plane. Their chromatic sum functions are, respectively,

$$f = f(x, y, z, t, \omega; \lambda) = \sum_{M \in \mathscr{M}} P(M; \lambda) x^{m(M)} y^{n(M)} z^{l(M)} t^{s(M)} \omega^{d(M)};$$

$$f_{\mathscr{P}} = f_{\mathscr{P}}(x, y, z, t, \omega; \lambda) = \sum_{M \in \mathscr{P}} P(M; \lambda) x^{m(M)} y^{n(M)} z^{l(M)} t^{s(M)} \omega^{d(M)},$$

where $m(M)$, $n(M)$, $l(M)$, $s(M)$ and $d(M)$ be, respectively, the valency of the root-vertex of M, the valency of the root-face of M, the number of edges of M, the number of nonroot-vertices of M and the number of nonroot-faces of M. $P(M; \lambda)$ is the chromatic polynomial of M.

Now two well-known formula on chromatic polynomials of maps should be mentioned for further use. The first one is

$$P(M; \lambda) = P(M - e; \lambda) - P(M \bullet e; \lambda) \tag{1}$$

for any map M, where e is an edge of M, $M - e$ and $M \bullet e$ stand for the maps obtained by deleting and contracting e from M, respectively. The second is

$$P(M_1 \cup M_2; \lambda) = \frac{1}{\lambda(\lambda - 1) \cdots (\lambda - i + 1)} P(M_1; \lambda) P(M_2; \lambda) \tag{2}$$

provided that $M_1 \cap M_2 = K_i$, the complete graph of order i, $i \geqslant 1$.

In this paper, for any map M, $e_r(M)$ stand for the root-edge of M.

2. Maps on the sphere

In this section, we will set up the equations satisfied by the chromatic sum functions of rooted general maps on the sphere. An edge is called double edge (a double edge on the plane is also called *isthmus* by some authors) if each side of it is on the boundary of the same face. The set $\mathscr{M}$ may be divided into four parts as

$$\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2 + \mathscr{M}_3 + \mathscr{M}_4, \tag{3}$$

where $\mathscr{M}_1$ consists of only the vertex map,

$$\mathscr{M}_2 = \{M | M \in \mathscr{M} - \mathscr{M}_1,\ e_r(M) \text{ is a loop}\};$$
$$\mathscr{M}_3 = \{M | M \in \mathscr{M} - \mathscr{M}_1,\ e_r(M) \text{ is a double edge}\};$$
$$\mathscr{M}_4 = \{M | M \in \mathscr{M} - \mathscr{M}_1 - \mathscr{M}_2 - \mathscr{M}_3\}.$$

Let f_i $(i=1, 2, 3, 4)$ be the chromatic sum function of $\mathscr{M}_i$. Let $f|_{x=1} = f(1, y, z, t, \omega; \lambda)$, $f|_{y=1} = f(x, 1, z, t, \omega; \lambda)$. For chromatic sum function f, we denote $\delta_\varphi f = (\varphi f - f|_{\varphi=1})/(\varphi - 1)$, here $\varphi = x$, or $\varphi = y$. Then

$$f_1 = \lambda; \quad f_2 = 0; \quad f_3 = \lambda^{-1}(\lambda - 1) x y^2 z t f f|_{x=1}. \tag{4}$$

It is easy to see that the following Lemmas hold.

Lemma 1. *Let* $\mathscr{M}_{\langle 4 \rangle} = \{M - e_r(M) | M \in \mathscr{M}_4\}$. *Then*

$$\mathscr{M}_{\langle 4 \rangle} \subseteq \mathscr{M}.$$

For a map M, let $\Delta_i(M)$ be the map obtained by adding a new edge from the root-vertex to the ith vertex on the root-face boundary, $i = 0, 1, 2, \ldots, n(M)$ ($n(M)$ is the valency of root-face of M). Let $\mathscr{M}^* = \sum_{M \in \mathscr{M}} \{\Delta_i(M) | n(M) \geqslant i \geqslant 0\}$. It is easily seen that $\mathscr{M}_4 \subseteq \mathscr{M}^*$.

Lemma 2. *For* $\mathscr{M}_4$, *we have*

$$\mathscr{M}_4 = \mathscr{M}^* - \mathscr{M}_2.$$