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Weak orders admitting a perpendicular linear order

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Abstract

Two orders on the same set are perpendicular if the constant maps and the identity map are the only maps preserving both orders. We characterize the finite weak orders admitting a perpendicular linear order.

MSC: 06A06; 06A07

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1. Introduction and presentation of the main results

An *order* on a set V is a reflexive, antisymmetric and transitive binary relation on V, specified as a set P of ordered pairs of members of V. Endowed with this order, V is a *partially ordered set* (*poset* for short) that we will denote simply by P if this causes no confusion. As usual, we denote by $x \leqslant y$ (or $x \leqslant_P y$) the fact that the pair (x, y) belongs to P. We denote by P^{d} the *dual* of P, that is, the order defined on V by $x \leqslant_{P^{\mathrm{d}}} y$ if and only if $y \leqslant_P x$. If X is a subset of V, then $P - X$ is the order induced by P on $V - X$. If $X = \{x\}$, then we use the notation $P - x$ instead of $P - \{x\}$.

Throughout this paper, all orders will be *finite*.

A map $f : V \to V$ *preserves* P, or is an *endomorphism* of P, if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in P$. We denote by P^P the set of all maps which preserve P; it contains the identity map and the constant maps, which are called *trivial*. In [1], Demetrovics et al. introduced the notion of *perpendicular orders* as a pair of orders on the same set sharing only the trivial endomorphisms. This notion arises naturally from a problem about maximal clones in universal algebra [6,8]. There are two basic results about perpendicular orders:

Theorem 1. (i) *Every linear order, having at least four elements, has a perpendicular linear order.*

(ii) *If* $q(n)$ *denotes the number of linear orders perpendicular to the natural order on* $\{1, \ldots, n\}$, *then* $\lim_{n\to+\infty} q(n)/n! = e^{-2} = 0.1353\ldots$.

Note that if L is a linear order on n elements, say $L = 1 < 2 < \cdots < n$, then every order is perpendicular to L if $n \leqslant 2$, no order at all is perpendicular to L if $n = 3$. For $n \geqslant 4$, the linear order $2 < 4 < \cdots < 2k < 1 < \cdots < 2k - 1$ where $n = 2k$ is perpendicular to L whereas if $n = 2k + 1$, the linear order $2 < 4 < \cdots < 2k < 1 < \cdots < 2k - 3 < 2k + 1 < 2k - 1$ is

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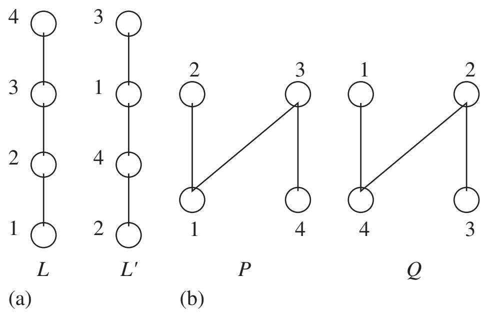

Fig. 1. (a) L and L' are perpendicular. (b) P is perpendicular to Q since $P = L \cap L'$ and $Q = L \cap L'^{\mathrm{d}}$.

perpendicular to L. Before stating the next result we recall the notion of an autonomous set. We say that a subset C of V is *autonomous* for P if

$$v < a \Rightarrow v < a' \quad \text{and} \quad a < v \Rightarrow a' < v$$

hold for all a, a' in C and v in $V - C$. Note that in the case of a linear order, autonomous sets are just the intervals. See [5] for more about autonomous sets.

Theorem 2. *Let L and L' be two distinct linear orders on the same set and let $P := L \cap L'$ and $Q := L \cap L'^{\mathrm{d}}$. The following properties are equivalent.*

(i) *P and Q are perpendicular*;
(ii) *L and L' are perpendicular*;
(iii) *L and L' have no non-trivial interval in common*;
(iv) *P has no non-trivial autonomous set*;
(v) *P and Q have no non-trivial autonomous set in common*.

Theorem 1 is due to Nozaki et al. [7] (see [11] for a new proof, based on a probabilistic argument). Theorem 2 gathers several results proved independently. Equivalence (i) $\Leftrightarrow$ (v) is due to Rival and Zaguia [9], equivalence (ii) $\Leftrightarrow$ (iii) to Nozaki et al. [7] and equivalence (iii) $\Leftrightarrow$ (iv) to the second author of the present paper [11]. For a direct proof of Theorem 2, obtain the equivalence (i) $\Leftrightarrow$ (ii) from the fact that $P^P \cap Q^Q = L^L \cap L'^{L'}$; next, use or prove the equivalences (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv) and observe that the implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) are trivial (Fig. 1).

The main result of this paper, Theorem 3, gives necessary and sufficient conditions for a weak order P to admit a perpendicular linear order L. Essentially, Theorem 3 says that such a linear order exists if and only if the levels of P are not "too big".

In order to state our main result, we give the notations and definitions we need.

We denote by $\mathrm{Min}(P)$, respectively $\mathrm{Max}(P)$, the set of minimal elements of V with respect to P, respectively, the set of maximal elements of V with respect to P. An element x of V is *extremal* in P if $x \in \mathrm{Min}(P) \cup \mathrm{Max}(P)$. We recall that the decomposition of an order P into *levels* is the sequence $P_0, \ldots, P_n, \ldots$ defined by induction by the formula

$$P_n := \mathrm{Min}(P - \cup\{P_{n'} : n' < n\}).$$

In particular, $P_0 = \mathrm{Min}(P)$.

The *height* of P, denoted by $h(P)$, is the least integer n such that $P_n = \emptyset$, or, equivalently, the number of levels. Hence, $V = \cup\{P_n : n < h(P)\}$. The height of P is also the number of vertices in a longest chain (total order) included in P [2]. In particular, P is *bipartite* if and only if $h(P) \leqslant 2$.

We denote by p the number of elements of V. In the sequel we suppose that $p \neq 0$ and (contrarily to the above definition), by a level we mean a level P_i with $0 \leqslant i < h(P)$. We denote by p_i the number of elements of P_i. We set $p_{-1} := p_{h(P)} := 1$.