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A generalization of Dirac's theorem on cycles through k vertices in k-connected graphs

Evelyne Flandrin[a], Hao Li[a], Antoni Marczyk[b], Mariusz Woźniak[b]

[a] L R I, UMR 8623, Bât. 490, Université de Paris-Sud, 91405 Orsay, France
[b] Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

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Abstract

Let X be a subset of the vertex set of a graph G. We denote by $\kappa(X)$ the smallest number of vertices separating two vertices of X if X does not induce a complete subgraph of G, otherwise we put $\kappa(X) = |X| - 1$ if $|X| \geqslant 2$ and $\kappa(X) = 1$ if $|X| = 1$. We prove that if $\kappa(X) \geqslant 2$ then every set of at most $\kappa(X)$ vertices of X is contained in a cycle of G. Thus, we generalize a similar result of Dirac. Applying this theorem we improve our previous result involving an Ore-type condition and give another proof of a slightly improved version of a theorem of Broersma et al.

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1. Introduction

Throughout this article we will consider only undirected, finite and simple graphs. Let G be a connected graph and let S be a proper subset of $V(G)$. S is called a *vertex cut* of G if the graph $G - S$ (i.e., the graph obtained by removing all vertices of S from G) is not connected. Let S be a vertex cut of G and Y, Z two connected components of $G - S$. If $x \in V(Y)$ and $y \in V(Z)$, we say that the vertex cut S *separates* x and y. Observe that only two nonadjacent vertices can be separated and the vertex cut S that separates x and y contains neither x nor y.

Let X be a subset of the vertex set of a graph G such that $|X| \geqslant 2$ and X does not induce a complete subgraph of G. The *connectivity of* X in G, denoted by $\kappa(X)$, is the smallest number k such that there exists in G a vertex cut of k vertices that separates two vertices of X. If X is a clique in G, then, by definition, $\kappa(X) = |X| - 1$ for $|X| \geqslant 2$ and $\kappa(X) = 1$ if $X = 1$. The number $\kappa(V(G)) = \kappa(G)$ is called the *connectivity* of the graph G. Thus, for the complete graph K_n we have $\kappa(K_n) = n - 1$, $n \geqslant 2$, and $\kappa(K_1) = 1$. Note that some authors (see for example [4]) define $\kappa(X) = \infty$ if X is a clique. By $\alpha(X)$ we denote the maximum number of pairwise nonadjacent vertices in the subgraph of G induced by X. We say that X is *cyclable* in G if G has a cycle containing all vertices of X.

The investigation on cycles passing through a given set of vertices in k-connected graphs was initiated by Dirac [7].

E-mail address: marczyk@uci.agh.edu.pl.

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Theorem 1. *Let G be a k-connected graph, where $k \geqslant 2$, and let X be a set of k vertices of G. Then there is a cycle in G containing every vertex of X.*

There are many improvements of the last theorem. For example, Egawa et al. [8] proved the common generalizations of Theorem 1 and the classical Dirac's theorem [6] on the existence of hamiltonian cycles in graphs.

Theorem 2. *Let G be a k-connected graph, where $k \geqslant 2$, and let X be a set of k vertices of G. Then G contains either a cycle of length at least $2\delta(G)$ including every vertex of X or a hamiltonian cycle.*

Recently Häggkvist and Mader [12] showed that every set of $k + \lfloor \frac{1}{3}\sqrt{k} \rfloor$ vertices in a k-connected k-regular graph belongs to some cycle.

Bollobás and Brightwell [1] and Shi [18] obtained an extension of Dirac's theorem on hamiltonian graphs.

Theorem 3. *Let G be a 2-connected graph of order n and let X be a set of vertices of G. If $d_G(x) \geqslant n/2$ for each $x \in X$, then X is cyclable.*

Shi [18] improved both Ore's theorem [16] and the previous one in the following way.

Theorem 4. *Let G be a graph of order n and let X be a subset of its vertex set such that $\kappa(X) \geqslant 2$. If $d_G(x) + d_G(y) \geqslant n$ for each pair x, y of nonadjacent vertices of X, then X is cyclable in G.*

This theorem was proved under the assumption that the graph is 2-connected. However, the presented version follows easily from a theorem due to Ota [17] that we give in Section 7.

The main result of this paper is the following generalization of Theorem 1 involving the notion of the connectivity of a set of vertices.

Theorem 5. *Let G be a graph and Y a subset of $V(G)$ with $\kappa(Y) \geqslant 2$. Let X be a subset of Y with $|X| \leqslant \kappa(Y)$. Then X is cyclable in G.*

Broersma et al. [4] studied cyclability of sets of vertices of graphs satisfying a local Chvátal–Erdős-type condition that involves the defined above parameters. They obtained a generalization of a result of Fournier [10] and of Chvátal–Erdős theorem under the assumption that the graph is 2-connected. The first application of Theorem 5 is an alternative proof of their theorem (there is a gap in the original proof).

Theorem 6. *Let G be a graph and let $X \subset V(G)$ with $\kappa(X) \geqslant 2$. If $\alpha(X) \leqslant \kappa(X)$, then X is cyclable in G.*

The second one is the following extension of a result of Flandrin et al. [9].

Theorem 7. *Let $G = (V, E)$ be a graph of order n. Let $X_1, X_2, \ldots, X_q$ be subsets of the vertex set V such that the union $X = X_1 \cup X_2 \cup \cdots \cup X_q$ satisfies $2 \leqslant q \leqslant \kappa(X)$. If for each i, $i = 1, 2, \ldots, q$, and for any pair of nonadjacent vertices $x, y \in X_i$, we have*

$$d(x) + d(y) \geqslant n,$$

then X is cyclable in G.

The condition of the last theorem is weaker than that of Shi and is called the *regional Ore's condition.*

As an immediate consequence of Theorem 7 we get the following generalization of Theorem 5.

Theorem 8. *Let $G = (V, E)$ be a graph, $k \geqslant 2$. For every set of k cliques $X_1, X_2, \ldots, X_k$ of G, such that $\kappa(X_1 \cup X_2 \cup \cdots \cup X_k) \geqslant k \geqslant 2$ there exists a cycle of G containing all vertices of these cliques.* □

Therefore, for any set of k cliques in a k-connected graph G there is a cycle of G containing all vertices of these cliques.

Let us recall the notion of k-closure of a graph which was introduced in the classical paper due to Bondy and Chvátal [2]. Namely, given an integer k, we will call the k-closure of G the graph obtained by recursively joining pairs x, y of