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Cubic maximal nontraceable graphs ☆

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Abstract

We determine a lower bound for the number of edges of a 2-connected maximal nontraceable graph, and present a construction of an infinite family of maximal nontraceable graphs that realize this bound.

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1. Introduction

We consider only simple, finite graphs G and denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. The *open neighbourhood* of a vertex v in G is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. If U is a nonempty subset of $V(G)$, then $\langle U \rangle$ denotes the subgraph of G induced by U.

A graph G is *hamiltonian* if it has a *hamiltonian cycle* (a cycle containing all the vertices of G), and *traceable* if it has a *hamiltonian path* (a path containing all the vertices of G). A graph G is *maximal nonhamiltonian* (MNH) if G is not hamiltonian, but $G+e$ is hamiltonian for each $e \in E(\overline{G})$, where $\overline{G}$ denotes the complement of G. A graph G is *maximal nontraceable* (MNT) if G is not traceable, but $G+e$ is traceable for each $e \in E(\overline{G})$. A graph G is *hypohamiltonian* if G is not hamiltonian, but every vertex-deleted subgraph $G-v$ of G is hamiltonian. We say that a graph G is *maximal hypohamiltonian* (MHH) if it is MNH and hypohamiltonian.

In 1978, Bollobás [1] posed the problem of finding the least number of edges, $f(n)$, in a MNH graph of order n. Bondy [2] had already shown that a MNH graph with order $n \geqslant 7$ that contained m vertices of degree 2 had at least $(3n+m)/2$ edges, and hence $f(n) \geqslant \lceil 3n/2 \rceil$ for $n \geqslant 7$. Combined results of Clark et al. [5,6] and Lin et al. [9] show that $f(n) = \lceil 3n/2 \rceil$ for $n \geqslant 19$ and for $n = 6, 10, 11, 12, 13, 17$. The values of $f(n)$ for the remaining values of n are also given in [9].

Let $g(n)$ be the minimum size of a MNT graph of order n. Dudek et al. [7] showed that $g(n) \geqslant (3n-20)/2$ for all n and, by means of a recursive construction, they found MNT graphs of order n and size $O(n \log n)$. To date, no cubic

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MNT graphs have been reported. We construct an infinite family of cubic MNT graphs, thus showing that $g(n) \leqslant 3n/2$ for infinitely many n.

Now let $g_2(n)$ be the minimum size of a 2-connected MNT graph of order n. We prove that $g_2(n) \geqslant \lceil 3n/2 \rceil$ for $n \geqslant 7$. It then follows from our constructions that $g_2(n) = \lceil 3n/2 \rceil$ for $n = 8p$ for $p \geqslant 5$, $n = 8p + 2$ for $p \geqslant 6$, $n = 8p + 4$ for $p = 3$ and $p \geq 6$, and $n = 8p + 6$ for $p \geqslant 4$.

2. A lower bound for the size of a 2-connected MNT graph

Bondy [2] proved that if G is a 2-connected MNH graph and $v \in V(G)$ with degree $d(v) = 2$, then each neighbour of v has degree at least 4. He also showed that the neighbours of such a vertex are in fact adjacent.

In order to prove a corresponding result for 2-connected MNT graphs we need the following result.

Lemma 2.1. *Let Q be a path in a MNT graph G. If $\langle V(Q) \rangle$ is not complete, then some internal vertex of Q has a neighbour in $G - V(Q)$.*

Proof. Let u and v be two nonadjacent vertices of $\langle V(Q) \rangle$. Then $G + uv$ has a hamiltonian path P. Let x and y be the two endvertices of Q and suppose no internal vertex of Q has a neighbour in $G - V(Q)$. Then P has a subpath R in $\langle V(Q) \rangle + uv$ and R has either one or both endvertices in $\{x, y\}$. If R has only one endvertex in $\{x, y\}$, then P has an endvertex in Q. In either case the path obtained from P by replacing R with Q is a hamiltonian path of G. □

Lemma 2.2. *If G is a MNT graph and $v \in V(G)$ with $d(v) = 2$, then the neighbours of v are adjacent. If in addition G is 2-connected, then each neighbour of v has degree at least 4.*

Proof. Let $N_G(v) = \{x_1, x_2\}$ and let Q be the path $x_1 v x_2$. Since $N_G(v) \subseteq Q$, it follows from Lemma 2.1 that $\langle V(Q) \rangle$ is a complete graph; hence x_1 and x_2 are adjacent.

Now assume that G is 2-connected. Since G is not traceable we assume $d(x_1) > 2$. Then also $d(x_2) > 2$ otherwise x_1 would be a cut vertex of G.

Let z be a neighbour of x_1 and let Q be the path $z x_1 v x_2$. Since $d(v) = 2$ the graph $\langle V(Q) \rangle$ is not complete, and hence it follows from Lemma 2.1 that x_1 has a neighbour in $G - V(Q)$. Thus $d(x_1) \geqslant 4$. Similarly $d(x_2) \geqslant 4$. □

We also have the following two lemmas concerning MNT graphs that have vertices of degree 2.

Lemma 2.3. *Suppose G is a 2-connected MNT graph. Suppose $v_1, v_2 \in V(G)$ such that $d(v_1) = d(v_2) = 2$ and v_1 and v_2 have exactly one common neighbour x. Then $d(x) \geqslant 5$.*

Proof. The vertices v_1 and v_2 cannot be adjacent otherwise x would be a cut vertex. Let $N(v_i) = \{x, y_i\}$; $i = 1, 2$. It follows from Lemma 2.2 that x is adjacent to y_i; $i = 1, 2$. Let Q be the path $y_1 v_1 x v_2 y_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 2.1 that x has a neighbour in $G - V(Q)$. Hence $d(x) \geqslant 5$. □

Lemma 2.4. *Suppose G is a MNT graph. Suppose $v_1, v_2 \in V(G)$ such that $d(v_1) = d(v_2) = 2$ and v_1 and v_2 have the same two neighbours x_1 and x_2. Then $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$. Also $d(x_1) = d(x_2) \geqslant 5$.*

Proof. From Lemma 2.2 it follows that x_1 and x_2 are adjacent. Let Q be the path $x_2 v_1 x_1 v_2$. $\langle V(Q) \rangle$ is not complete since v_1 and v_2 are not adjacent. Thus it follows from Lemma 2.1 that x_1 has a neighbour in $G - V(Q)$. Now suppose $p \in N_{G-V(Q)}(x_1)$ and $p \notin N_G(x_2)$. Then a hamiltonian path P in $G + px_2$ contains a subpath of either of the forms given in the first column of Table 1. Note that $i, j \in \{1, 2\}$; $i \neq j$ and that L represents a subpath of P in $G - \{x_1, x_2, v_1, v_2, p\}$. If each of the subpaths is replaced by the corresponding subpath in the second column of the table we obtain a hamiltonian path P' in G, which leads to a contradiction.

Hence $p \in N_G(x_2)$. Thus $N_G(x_1) - \{x_2\} \subseteq N_G(x_2) - \{x_1\}$. Similarly $N_G(x_2) - \{x_1\} \subseteq N_G(x_1) - \{x_2\}$. Thus $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and hence $d(x_1) = d(x_2)$. Now let Q be the path $p x_1 v_1 x_2 v_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 2.1 that x_1 or x_2 has a neighbour in $G - V(Q)$. Hence $d(x_1) = d(x_2) \geqslant 5$. □

We now consider the size of a 2-connected MNT graph.