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On a cycle through a specified linear forest of a graph

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Abstract

Results on the existence of a cycle containing a given linear forest are proved.

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1. Introduction and results

We use [3] for terminology and notation not defined here and consider finite simple graphs only. Let G be a graph, $X \subseteq V(G)$, and $G[X]$ be the subgraph of G induced by X. For $S \subset V(G)$ let $\omega_X(G-S)$ be the number of components of the graph $G-S$, which contain a vertex of X. Let $\kappa(X)$ be infinity if $G[X]$ is complete or the minimum cardinality of a set $S \subset V(G)$ with $\omega_X(G-S) \geqslant 2$. A *linear forest* is an acyclic graph of maximum degree at most 2. Given a linear forest L (as a subgraph of G) with its vertex set $V(L)$, a cycle of G containing $V(L)$ and all edges of L is an *L-cycle*. Theorems 1 and 2 are consequences of results in [2,4] and in [6,11], respectively.

Theorem 1 (*Broersma et al. [2], Dirac [4]*). *Let G be a graph, $X \subseteq V(G)$, $|X| \leqslant \kappa(X)$, and $\kappa(X) \geqslant 2$. Then G has an X-cycle.*

Given $t > 0$, X is called *t-tough in G if* $\omega_X(G-S) = 1$ or $\omega_X(G-S) \leqslant |S|/t$ for all $S \subset V(G)$. Let $\tau_X(G)$ be the maximum value of t for which X is t-tough in G. If $G[X]$ is complete we define $\tau_X(G) = \infty$.

Theorem 2 (*Harant [6], Watkins and Mesner [11]*). *Let G be a graph, $X \subseteq V(G)$, $|X| \leqslant \kappa(X) + 1$, and $\kappa(X) \geqslant 3$. If $\tau_X(G) \geqslant 1$, then G has an X-cycle.*

The following theorem was proved in [5].

Theorem 3 (*Häggkvist and Thomassen [5]*). *If Y is a set of independent edges of a graph G with $|Y| \leqslant \kappa(V(Y)) - 1$ then G has a Y-cycle.*

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In [8] Kawarabayashi gave the outline of a proof of the Lovász–Woodall–Conjecture (see [9,12]) concerning the existence of a cycle through $\kappa(V(G))$ independent edges of a graph G and referred to forthcoming papers.

Theorem 4 (*Kawarabayashi [8]*). *If Y is a set of independent edges of a graph G with $|Y| \leqslant \kappa(V(G))$ and $\kappa(V(G)) \geqslant 2$ then G has a Y-cycle unless $\kappa(V(G))$ is odd and $G - Y$ is disconnected.*

For a path P let $e(P)$ be the number of edges of P and for a path set $\mathscr{P}$ we define $e(\mathscr{P}) = \sum_{P \in \mathscr{P}} e(P)$.

Theorem 5 (*Harant [6]*). *Let G be a graph, $X \subseteq V(G)$, $\mathscr{P}$ be a set of disjoint paths of length at least one in $G - X$, $|X| + e(\mathscr{P}) \leqslant \kappa(X \cup V(\mathscr{P}))$, and $\kappa(X \cup V(\mathscr{P})) \geqslant 2$. If G has a $\mathscr{P}$-cycle then G has an $(X \cup \mathscr{P})$-cycle.*

Results on the length of an $(X \cup \mathscr{P})$-cycle in case $|X| + e(\mathscr{P}) \leqslant \kappa(V(G)) - 2$ can be found in [7]. Theorem 6 is a consequence of Theorems 3 and 5. We give a short proof here. Theorem 7 is our main result.

Theorem 6. *Let G be a graph, $\emptyset \neq X \subseteq V(G)$, $\mathscr{P}$ be a set of disjoint paths of length at least one in $G - X$, and $\kappa(X \cup V(\mathscr{P})) \geqslant 2$. If $|X| + e(\mathscr{P}) \leqslant \kappa(X \cup V(\mathscr{P}))$ then G has an $(X \cup \mathscr{P})$-cycle.*

Theorem 7. *Let G be a graph, $X \subseteq V(G)$ with $|X| \geqslant 4$, and $\mathscr{P}$ be a nonempty set of disjoint paths of length at least one in $G - X$. If $|X| + e(\mathscr{P}) \leqslant \kappa(X \cup V(\mathscr{P})) + 1$ and $\tau_X(G) > \kappa(X \cup V(\mathscr{P}))/|X|$ then G has an $(X \cup \mathscr{P})$-cycle.*

2. Proofs

For $A, B \subseteq V(G)$ an *$A - B$-path* is a path P between A and B such that $|V(P) \cap A| = |V(P) \cap B| = 1$. A common vertex of A and B is also an $A - B$-path. A set $S \subseteq V(G)$ *separates A and B* if any $A - B$-path contains a vertex in S. Let $N(v)$ be the neighbourhood of $v \in V(G)$. Without mentioning in each case, we shall use the following properties.

(π_1) *A is t-tough if B is t-tough for $A \subseteq B \subseteq V(G)$.*
(π_2) *$\kappa(A) \geqslant \kappa(B)$ if $A \subseteq B \subseteq V(G)$.*
(π_3) *Let $A, B, B' \subseteq V(G)$ such that $B' \subseteq B$. If $S \subseteq V(G)$ separates A and B then S also separates A and B'.*
(π_4) *Let $a \in A \subseteq V(G)$ and $\kappa(A) < \infty$. Then $|N(a)| \geqslant \kappa(A)$ or $A \subseteq \{a\} \cup N(a)$.*
(π_5) *Let $A \subset V(G)$ and $b \in V(G) \backslash A$. If $|A| \geqslant \kappa(A \cup \{b\})$ then A and $N(b)$ cannot be separated by a set of at most $\kappa(A \cup \{b\}) - 1$ vertices.*

Let $i(P)$ be the number of inner vertices of a path P and $i(\mathscr{P}) = \sum_{P \in \mathscr{P}} i(P)$ for a path set $\mathscr{P}$. Obviously, $|\mathscr{P}| + i(\mathscr{P}) = e(\mathscr{P})$. In the sequel we shall write $i_G(.)$ and $\kappa_G(.)$ instead of $i(.)$ and $\kappa(.)$, respectively, if it is important to distinguish in which graph G these values are calculated.

2.1. *Proof of Theorem 6*

Let G be a graph, $X \subseteq V(G)$, and $\mathscr{P}$ be a set of disjoint paths of $G - X$ each containing at least one edge with $|X| \geqslant 1$ and $|X| + |\mathscr{P}| \leqslant \kappa_G(X \cup V(\mathscr{P})) - i_G(\mathscr{P})$. Because of Theorem 1 we may assume $\mathscr{P} \neq \emptyset$. The proof is by induction on $i_G(\mathscr{P})$. For $i_G(\mathscr{P}) = 0$ we are done with $|\mathscr{P}| < |X| + |\mathscr{P}| \leqslant \kappa_G(X \cup V(\mathscr{P})) \leqslant \kappa_G(V(\mathscr{P}))$ and by using Theorems 3 and 5. Hence, we may assume $i_G(\mathscr{P}) \geqslant 1$. Let a be an inner vertex of a path $P \in \mathscr{P}$ and $N(a) \cap V(P) = \{b, c\}$. The graph G', the path P', and the set $\mathscr{P}'$ are defined by $V(G') = V(G) \backslash \{a\}$, $E(G') = E(G[V(G) \backslash \{a\}]) \cup \{bc\}$, $V(P') = V(P) \backslash \{a\}$, $E(P') = E(G[P \backslash \{a\}]) \cup \{bc\}$, and $\mathscr{P}' = (\mathscr{P} \backslash \{P\}) \cup \{P'\}$. Clearly, $\kappa_{G'}(X \cup V(\mathscr{P}')) \geqslant \kappa_G(X \cup V(\mathscr{P})) - 1$, $i_{G'}(\mathscr{P}') = i_G(\mathscr{P}) - 1$, and $|X| + |\mathscr{P}'| = |X| + |\mathscr{P}| \leqslant \kappa_G(X \cup V(\mathscr{P})) - i_G(\mathscr{P}) \leqslant \kappa_{G'}(X \cup V(\mathscr{P}')) - i_{G'}(\mathscr{P}')$. Hence, by the induction hypothesis there exists an $(X \cup \mathscr{P}')$-cycle C' of G' with $bc \in E(C')$. The cycle obtained from C' by replacing the edge bc by the path bac is an $(X \cup \mathscr{P})$-cycle of G. □

2.2. *Proof of Theorem 7*

A more detailed version of Menger's Theorem (see [10]) is the following lemma.