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Discrete Mathematics 307 (2007) 916–922
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

$[r, s, t]$-Chromatic numbers and hereditary properties of graphs

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Received 19 November 2003; received in revised form 5 March 2004; accepted 22 November 2005
Available online 10 October 2006

Abstract

Given non-negative integers r, s, and t, an $[r, s, t]$-*coloring* of a graph $G = (V(G), E(G))$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k-1\}$, $k \in \mathbb{N}$, such that $|c(v_i) - c(v_j)| \geqslant r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geqslant s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geqslant t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring.

We characterize the properties $\mathcal{O}(r, s, t, k) = \{G : \chi_{r,s,t}(G) \leqslant k\}$ for $k = 1, 2, 3$ as well as for $k \geqslant 3$ and $\max\{r, s, t\} = 1$ using well-known hereditary properties. The main results for $k \geqslant 3$ are summarized in a diagram.

MSC: 05C15

Keywords: $[r, s, t]$-Chromatic number; Hereditary properties; Generalized colorings

1. Introduction

Given non-negative integers r, s, and t with $\max\{r, s, t\} \geqslant 1$, an $[r, s, t]$-*coloring* of a finite and simple graph G with vertex set $V(G)$ and edge set $E(G)$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k-1\}$, $k \in \mathbb{N}$, such that $\left|c(v_i) - c(v_j)\right| \geqslant r$ for every two adjacent vertices v_i, v_j, $\left|c(e_i) - c(e_j)\right| \geqslant s$ for every two adjacent edges e_i, e_j, and $\left|c(v_i) - c(e_j)\right| \geqslant t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r, s, t]$-coloring.

This is an obvious generalization of the classical graph colorings since a $[1, 0, 0]$-coloring is an ordinary vertex coloring, a $[0, 1, 0]$-coloring is an edge coloring, and a $[1, 1, 1]$-coloring is a total coloring. First results on this kind of graph coloring can be found in [8].

Let $\mathcal{I}$ be the set of all finite, simple graphs. A *graph property* $\mathcal{P}$ is any isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ imply $H \in \mathcal{P}$. Analogously, a property $\mathcal{P}$ is called *induced hereditary* if for a graph G with the property $\mathcal{P}$ all of its induced subgraphs have the property $\mathcal{P}$. For surveys of hereditary properties of graphs see, e.g., [3,4,11].

Every hereditary property $\mathcal{P}$ is determined by the set of *minimal forbidden subgraphs* $F(\mathcal{P}) = \{G \in \overline{\mathcal{P}}$: every proper subgraph of G is in $\mathcal{P}\}$.

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doi:10.1016/j.disc.2005.11.055

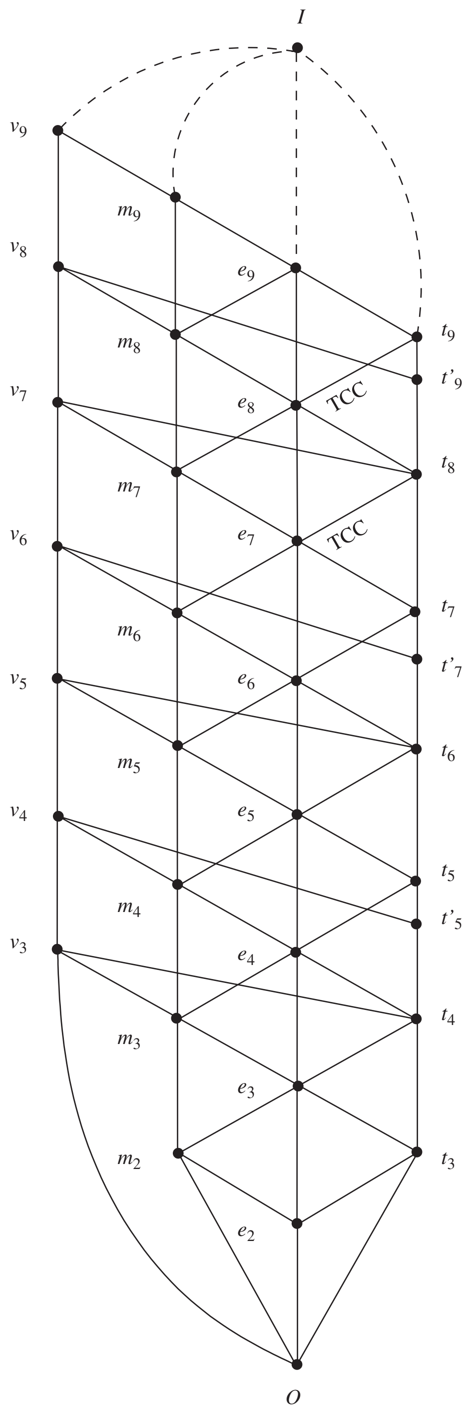

Fig. 1. Properties $\mathcal{O}(r, s, t, k)$ with $\max\{r, s, t\} \leqslant 1$ ($v_k = \mathcal{O}(1, 0, 1, k)$, $m_k = \mathcal{O}(1, 1, 0, k)$, $e_k = \mathcal{O}(0, 1, 1, k)$, $t_k = \mathcal{O}(1, 1, 1, k)$, $t'_k = t_k \cap \mathcal{I}_{k-2}$, $O = \mathcal{O}$, $I = \mathcal{I}$).

In this paper we investigate the properties $\mathcal{O}(r, s, t, k) = \{G \in \mathcal{I} : \chi_{r,s,t}(G) \leqslant k\}$ and the sets $F(\mathcal{O}(r, s, t, k)) = \{G \in \mathcal{I} : \chi_{r,s,t}(G) \geqslant k + 1$ and $\chi_{r,s,t}(H) \leqslant k$ for all proper subgraphs $H \subset G\}$ (Fig. 1).

Since $\chi_{r,s,t}(H) \leqslant \chi_{r,s,t}(G)$ if H is a subgraph of G, properties $\mathcal{O}(r, s, t, k)$ are hereditary and induced hereditary and since $\chi_{r,s,t}(C_i) \leqslant k$ for all components C_i of G implies $\chi_{r,s,t}(G) \leqslant k$, $\mathcal{O}(r, s, t, k)$ are also additive properties.