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BCH codes and distance multi- or fractional colorings in hypercubes asymptotically

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Abstract

The main result is a short and elementary proof for the author's exact asymptotic results on distance chromatic parameters (both number and index) in hypercubes. Moreover, the results are extended to those on fractional distance chromatic parameters and on distance multi-colorings. Inspiration comes from radio frequencies allocation problem. The basic idea is the observation that binary primitive narrow-sense BCH codes or their shortenings have size asymptotically within a constant factor below the largest possible size, $A(n, d)$, among all binary codes of the same length, n, and the same minimum distance, d, as $n \to \infty$ while d is constant. Also a lower bound in terms of $A(n, d)$ is obtained for $B(n, d)$, the largest size among linear binary codes.

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1. Introduction

BCH codes are important error-correcting codes which are linear and cyclic, and named after Bose and Chaudhuri [4,5] and Hocquenghem [15], the discoverers of the binary BCH codes. The aim of this note is to present a simple and elementary proof that binary BCH codes are nearly optimal in a rather peripheral case when their minimum distance d is fixed and the length n tends to infinity, which makes the long codes useless for detection of transmission errors. As is shown in [31], those codes appear sufficient for determining the exact order of growth of classical parameters in vertex and edge distance colorings of growing hypercubes. This asymptotics is extended to distance multi-colorings and to distance fractional chromatic parameters. Relations with applications to the frequency assignment in cellular telecommunication are recalled. Results on coding that we refer to in what follows can be found in the monograph by MacWilliams and Sloane [23] or books by Hankerson et al. [13], van Lint [22], or by Roman [29]. We refer: to the author's paper [31] for distance colorings; to the monograph by Scheinerman and Ullman [30] for fractional colorings and multi-colorings; and to [7] for a general overview of the frequency assignment.

We now recall some notation necessary for stating our main results. Given a graph G (G being loopless with multiple edges allowed), let $\chi(G)$ and $q(G)$ be, respectively, the chromatic number and the chromatic index of G. Let $L(G)$ stand for the line graph of G. Then $q(G) = \chi(L(G))$. Given a positive integer d, let G^d denote the dth power of G,

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G^d being the spanning supergraph of G with any two vertices being adjacent whenever their distance in G is between 1 and d inclusive. Hence $G = G^1$. Following [31], define the d^+*-distance chromatic* parameters: *number*, $\chi_d(G)$, and *index*, $q_d(G)$, of G as follows.

$$\chi_d(G) = \chi(G^d), \quad q_d(G) = \chi_d(L(G)) = \chi(L(G)^d). \tag{1}$$

The *b-fold chromatic number* of G, denoted $\chi^b(G)$, is defined to be the least cardinality a of a set of colors such that vertices can be colored with b-subsets of colors so that adjacent vertices are assigned disjoint subsets. Then $q^b(G) := \chi^b(L(G))$ is the *b-fold chromatic index* of G. Thus, $\chi^1 = \chi$ and $q^1 = q$. Then the *fractional chromatic* parameter, say π_{f} with $\pi = \chi$ or q, namely *number* $\chi_{\mathrm{f}}(G)$ and *index* $q_{\mathrm{f}}(G)$, is defined to be

$$\pi_{\mathrm{f}}(G) = \lim_{b\to\infty} \frac{\pi^b(G)}{b} = \inf_b \frac{\pi^b(G)}{b}, \tag{2}$$

where $\pi = \chi$ or q. If π therein is replaced by the corresponding d^+-distance parameter π_d then the notions $\pi_d^b(G)$ (or $\pi_{\mathrm{f},d}(G)$) are defined and they are called the d^+*-distance b-fold* (or resp. d^+*-distance fractional*) *chromatic* parameter: *number*, $\chi_d^b(G)$ or $\chi_{\mathrm{f},d}(G)$, if $\pi = \chi$; and *index*, $q_d^b(G)$ or $q_{\mathrm{f},d}(G)$, if $\pi = q$.

Remark 1.1. Each equality in (1) remains valid if on both sides of the equality the same superscript b or subscript f is added to the chromatic parameter therein, to χ or q (or to both if applicable).

The symbol Θ indicates the exact order of asymptotic growth. Recall that the floor $\lfloor x \rfloor$ of a real number x is the integer part of x. Moreover, the ceiling $\lceil x \rceil = -\lfloor -x \rfloor$.

Theorem 1. *Let Q_t be the t-cube. Then for any constants $d, b \in \mathbb{N}$, d being a constant distance bound, if the dimension $t \to \infty$ then all the three following distance chromatic parameters with the same $\pi = \chi$ or q have the same order of growth, namely*

$$\chi_d(Q_t), \quad \chi_{\mathrm{f},d}(Q_t), \quad \chi_d^b(Q_t)/b = \Theta(t^{\lfloor d/2 \rfloor}),$$
$$q_d(Q_t), \quad q_{\mathrm{f},d}(Q_t), \quad q_d^b(Q_t)/b = \Theta(t^{\lfloor (d+1)/2 \rfloor}).$$

Main part of the above theorem (concerning parameters χ_d and q_d only, together with good bounds on them) was proved in the author's paper [31]. A simplified thorough proof will be given in what follows. As a by-product, relatively good bounds on all of the above chromatic parameters have been obtained, see Theorems 15 and 16.

Recall the following notation from the theory of error-correcting codes. Let $A(n, d)$ be the maximum size (= the number of codewords) in any binary code (possibly nonlinear) of length n and minimum (Hamming) distance d. Let $B(n, d)$ be the corresponding maximum size among *linear* codes with the same parameters n and d.

Moreover, let $(2n)!!$ stand for $2^n n!$. We are going to use the asymptotic inequality $\gtrsim$ in the following sense. Write $a(n) \gtrsim c(n)$ whenever there exists an expression $b(n)$ such that $a(n) \geqslant b(n) \sim c(n)$ as $n \to \infty$. Our main result on binary codes follows.

Theorem 2.

$$B(n, d) \gtrsim \frac{A(n, d)}{(2\lfloor (d-1)/2 \rfloor)!!}$$

as $n \to \infty$ and d is constant.

2. On coding and BCH codes

Some facts from coding theory are recalled in order to keep the paper self-contained. We restrict our attention to binary codes only. Thus, a code of length n is a subset of the linear space $\{0, 1\}^n$ over Galois field GF(2), the code being linear if it is a linear subspace. The name (n, M, d) code stands for a code of length n, size (or cardinality) M, and minimum distance d. A linear (n, M, d) code $\mathscr{C}$ has size $M = 2^k$ and is named an $[n, k, d]$ code, where $k = \log_2 M$ is the *dimension* of $\mathscr{C}$.