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Longest cycles in almost regular 3-partite tournaments

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Abstract

If D is a digraph, then we denote by $V(D)$ its vertex set. A multipartite or c-partite tournament is an orientation of a complete c-partite graph. The global irregularity of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) \mid x, y \in V(D)\}.$$

If $i_g(D)=0$, then D is *regular*, and if $i_g(D) \leqslant 1$, then D is called almost regular. In 1997, Yeo has shown that each regular multipartite tournament is Hamiltonian. This remains valid for almost all almost regular c-partite tournaments with $c \geqslant 4$. However, there exist infinite families of almost regular 3-partite tournaments without any Hamiltonian cycle. In this paper we will prove that every vertex of an almost regular 3-partite tournament D is contained in a directed cycle of length at least $|V(D)| - 2$. Examples will show that this result is best possible.

Keywords: Multipartite tournaments; Almost regular multipartite tournaments; 3-Partite tournaments; Longest cycle

1. Terminology

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. By a *cycle* (*path*) we mean a directed cycle (directed path).

In this paper all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the property that there is no arc from Y to X.

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x, respectively. Let

$$\delta^-(D) = \delta^- = \min\{d^-(x) \mid x \in V(D)\},$$

$$\delta^+(D) = \delta^+ = \min\{d^+(x) \mid x \in V(D)\},$$

and $\delta(D) = \delta = \min\{\delta^+, \delta^-\}$ be the *minimum degree* of the digraph D.

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The *global irregularity* of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) \mid x, y \in V(D)\},$$

and the *local irregularity* by

$$i_l(D) = \max_{x \in V(D)} |d^+(x) - d^-(x)|.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leqslant 1$, then D is called *almost regular*.

A cycle of length m is an *m-cycle*. A cycle in a digraph D is *Hamiltonian* if it includes all the vertices of D. A *cycle-factor* of a digraph D is a spanning subdigraph of D consisting of disjoint cycles.

For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. A set $X \subseteq V(D)$ of vertices is *independent* if the induced subdigraph $D[X]$ has no arcs. The *independence number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D.

A digraph D is *strongly connected* or *strong* if, for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected.

2. Introduction and preliminary results

In 2002, Tewes et al. [4] have made the following observation.

Lemma 2.1 (*Tewes et al. [4]*). *If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$, then $||V_i| - |V_j|| \leqslant 2i_g(D)$ for $1 \leqslant i \leqslant j \leqslant c$.*

The next result is a part of the main theorem in Yeo's paper [9].

Theorem 2.2 (*Yeo [9]*). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c|$. If*

$$i_g(D) \leqslant \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

In the case that D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c| = r$, it follows from Lemma 2.1 that $|V_1| \geqslant r - 2$. If D is an almost regular c-partite tournament with $c \geqslant 4$, then Theorem 2.2 shows that D is Hamiltonian with exception of the case that $1 = |V_1| = |V_2| < |V_3| = 2 < |V_4| = 3$. In this case there really exists the following example which is not Hamiltonian.

Example 2.3. Let $V_1 = \{x\}$, $V_2 = \{y\}$, $V_3 = \{u_1, u_2\}$, $V_4 = \{v_1, v_2, v_3\}$ be the partite sets of a 4-partite tournament such that $u_1 \to \{x, y\} \to u_2 \to V_4 \to u_1$, $x \to y \to \{v_2, v_3\} \to x \to v_1 \to y$. Then it is straightforward to verify that D is almost regular without a Hamiltonian cycle.

We have shown above that almost all almost regular c-partite tournaments with $c \geqslant 4$ are Hamiltonian. However, the next examples will show that this is no longer true for almost regular 3-partite tournaments.

Example 2.4. Let D be a 3-partite tournament with the partite sets V_1, V_2, V_3 such that $|V_1| = r - 1$ and $|V_2| = |V_3| = r$ and $V_1 \to V_2 \to V_3 \to V_1$. Then D is almost regular without a Hamiltonian cycle.

The longest cycle in this infinite family of almost regular 3-partite tournaments has length $|V(D)| - 2$. In this paper we will prove that every vertex of an almost regular 3-partite tournament D is contained in a directed cycle of length at least $|V(D)| - 2$. In special cases we can even give better bounds.