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Distance irredundance and connected domination numbers of a graph☆

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Abstract

Let k be a positive integer and G be a connected graph. This paper considers the relations among four graph theoretical parameters: the k-domination number $\gamma_k(G)$, the connected k-domination number $\gamma_k^{\mathrm{c}}(G)$; the k-independent domination number $\gamma_k^{\mathrm{i}}(G)$ and the k-irredundance number $ir_k(G)$. The authors prove that if an ir_k-set X is a k-independent set of G, then $ir_k(G) = \gamma_k(G) = \gamma_k^{\mathrm{i}}(G)$, and that for $k \geqslant 2$, $\gamma_k^{\mathrm{c}}(G) = 1$ if $ir_k(G) = 1$, $\gamma_k^{\mathrm{c}}(G) \leqslant \max\{(2k+1)ir_k(G) - 2k, \frac{5}{2}ir_k(G)k - \frac{7}{2}k + 2\}$ if $ir_k(G)$ is odd, and $\gamma_k^{\mathrm{c}}(G) \leqslant \frac{5}{2}ir_k(G)k - 3k + 2$ if $ir_k(G)$ is even, which generalize some known results.

MSC: 05C69

Keywords: k-Irredundance number; k-Domination number; Connected k-domination number; k-Independent domination number; k-Independent set

1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [5] or [15]. Let $G = (V, E)$ be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. The distance $d_G(x, y)$ between two vertices x and y is the length of a shortest xy-path in G. Let k be a positive integer. For every vertex $x \in V(G)$, the open k-neighborhood $N_k(x)$ of x is defined as $N_k(x) = \{y \in V(G) : \ d_G(x, y) \leqslant k, x \neq y\}$. The closed k-neighborhood $N_k[x]$ of x in G is defined as $N_k(x) \cup \{x\}$. Likewise, one may define the open (resp. closed) k-neighborhood of a set X of vertices in $V(G)$, denoted by $N_k(X)$ (resp. $N_k[X]$), as the union of the open (resp. closed) k-neighborhood $N_k(x)$ (resp. $N_k[x]$) of vertices in X.

For $x \in X$, we use $I_k(x \in X)$ to denote the set of vertices of G that are in $N_k[x]$ but not in $N_k[X - \{x\}]$. If $I_k(x \in X) = \emptyset$, then x is said to be k-redundant in X. In the context of a communication network, this means that any vertex that may receive communications from X within distance k may also be informed from $X - \{x\}$ within distance k. A set X containing no k-redundant vertex is called k-irredundant, that is, $I_k(x \in X) \neq \emptyset$ for any $x \in X$.

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A k-irredundant set X of G is called to be maximal if $X \cup \{y\}$ is not a k-irredundant set of G for any $y \in V(G) - X$. The k-irredundance number of G, denoted by $ir_k(G)$, is the minimum cardinality taken over all maximal k-irredundant sets of G. A maximal k-irredundant set of cardinality $ir_k(G)$ is called an ir_k-set. The concept of the k-irredundance was introduced by Hattingh and Henning [9].

A set D of vertices in G is called to be a k-dominating set of G if every vertex of $V(G) - D$ is within distance k from some vertex of D. A k-dominating set D is called to be connected if $G[D]$, a subgraph of G induced by D, is connected. The minimum cardinality among all k-dominating sets (resp. connected k-dominating sets) of G is called the k-domination number (resp. connected k-domination number) of G and is denoted by $\gamma_k(G)$ (resp. $\gamma_k^{\mathrm{c}}(G)$). The concept of the k-dominating set was introduced by Chang and Nemhauser [6,7].

A set I of vertices of G is said to be a k-independent set if every vertex in I is at distance at least $k+1$ from every other vertex of I in G; and a k-independent dominating set if I is a k-independent set and a k-dominating set, or equivalently, is a maximal k-independent set. The k-independent number $\alpha_k(G)$ is defined as the maximum cardinality taken over all k-independent sets of G; the k-independent domination number $\gamma_k^{\mathrm{i}}(G)$ is defined as the minimum cardinality taken over all k-independent dominating sets of G.

Since the distance versions of domination have a strong background of applications, many efforts have been made by several authors to establish the relationships among distance parameters (see, for example [6,7,9–14]). It is quite difficult to determine the value of $\gamma_k(G)$ or $\gamma_k^{\mathrm{c}}(G)$ for any given graph G. However, from definitions, it is clear that $ir_k(G) \leqslant \gamma_k(G) \leqslant \gamma_k^{\mathrm{i}}(G)$ for any graph G. For $k=1$, Allan et al. [2] proved that if an ir_1-set X is an independent set of G, then $ir_1(G) = \gamma_1(G) = \gamma_1^{\mathrm{i}}(G)$. Recently, Li [13] has established an upper bound of $\gamma_k^{\mathrm{c}}(G)$ in terms of other graph theoretical parameter. For $k=1$, Allan and Laskar [1], independently, Bollobas and Cockayne [4] established $\gamma_1(G) \leqslant 2ir_1(G) - 1$, which is extended to $\gamma_k(G) \leqslant 2ir_k(G) - 1$ by Hattingh and Henning [9]; Bo and Liu [3] established $\gamma_1^{\mathrm{c}}(G) \leqslant 3ir_1(G) - 2$.

In this paper, we prove that if an ir_k-set X is a k-independent set of G then $ir_k(G) = \gamma_k(G) = \gamma_k^{\mathrm{i}}(G)$, and that, for a connected graph G and $k \geqslant 2$, $\gamma_k^{\mathrm{c}}(G) = 1$ if $ir_k(G) = 1$, $\gamma_k^{\mathrm{c}}(G) \leqslant \max\{(2k+1)ir_k(G) - 2k, \frac{5}{2}ir_k(G)k - \frac{7}{2}k + 2\}$ if $ir_k(G)$ is odd, and $\gamma_k^{\mathrm{c}}(G) \leqslant \frac{5}{2}ir_k(G)k - 3k + 2$ if $ir_k(G)$ is even, and these bounds are best possible. The former generalizes Allan et al.'s result and the latter generalizes Bo and Liu's result. As a byproduct of the proof of our main result, we also obtain $\gamma_k(G) \leqslant 2ir_k(G) - 1$.

The proofs of our main results are in Section 3 and some lemmas are given in Section 2.

2. Lemmas

A k-independent set I of G is called to be maximal if $X \cup \{y\}$ is not a k-independent set of G for any $y \in V(G) - X$. A k-dominating set D of G is called to be minimal if $D - \{x\}$ is not a k-dominating set of G for any $x \in D$.

Lemma 2.1. *Let I be a k-independent set of G. Then I is maximal if and only if I is a minimal k-dominating set*, *whereby $\gamma_k^{\mathrm{i}}(G) \leqslant \alpha_k(G)$.*

Proof. Since I is a k-independent set of G, by definition, I is maximal if and only if every vertex of $V(G) - I$ is within distance k from some vertex of I and the distance between two vertices in I is larger than k or, equivalently, I is a minimal k-dominating set of G. □

The special case for $k=1$ of the following result is due to Cockayne and Hedetniemi [8], and it is stated in Lemma 2 in [9] without proof.

Lemma 2.2. *Let D be a k-dominating set of G. Then D is minimal if and only if D is a maximal k-irredundant set*, *whereby $ir_k(G) \leqslant \gamma_k(G)$.*

Proof. Since D is a k-dominating set of G, by definition, D is minimal if and only if every vertex of $V(G) - D$ is within distance k from some vertex of D and the removal of any vertex $x \in D$ results in a vertex y in $V(G) - (D - \{x\})$ at distance greater than k from every vertex in $D - \{x\}$ or, equivalently, $I_k(x \in D) \neq \emptyset$ for any $x \in D$ but there is some $x \in D' = D \cup \{y\}$ such that $I_k(x \in D') = \emptyset$ for any $y \in V(G) - D$, that is, D is a maximal k-irredundant set of G. □