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Existence of r-self-orthogonal Latin squares ☆

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Abstract

Two Latin squares of order v are r-orthogonal if their superposition produces exactly r distinct ordered pairs. If the second square is the transpose of the first one, we say that the first square is r-self-orthogonal, denoted by r-SOLS(v). It has been proved that for any integer $v \geqslant 28$, there exists an r-SOLS(v) if and only if $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$. In this paper, we give an almost complete solution for the existence of r-self-orthogonal Latin squares.

Keywords: Latin square; r-Orthogonal; r-Self-orthogonal; Transversal

1. Introduction

Two Latin squares of order v, $L=(l_{ij})$ and $M=(m_{ij})$, are said to be *r-orthogonal* if their superposition produces exactly r distinct pairs, that is

$$|\{(l_{ij}, m_{ij}) : 0 \leqslant i, j \leqslant v-1\}| = r.$$

Belyavskaya (see [2–4]) first systematically treated the following question: for which integers v and r does a pair of r-orthogonal Latin squares of order v exist? Evidently, $v \leqslant r \leqslant v^2$, and an easy argument establishes that $r \notin \{v+1, v^2-1\}$. In papers by Colbourn and Zhu [8], Zhu and Zhang [19,20], this question has been completely answered, and the final result is in the following theorem.

Theorem 1.1 (*Zhu and Zhang [20, Theorem 2.1]*). *For any integer $v \geqslant 2$, there exists a pair of r-orthogonal Latin squares of order v if and only if $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$ with the exceptions of v and r shown in Table 1.*

In a pair of r-orthogonal Latin squares of order v, if the second square is the transpose of the first one, we say that the first square is *r-self-orthogonal*, denoted by r-SOLS(v). When an r-SOLS(v) exists, we can simply list only one square for a pair of r-orthogonal Latin squares of order v.

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Table 1

order v	Genuine exceptions of r
2	4
3	5, 6, 7
4	7, 10, 11, 13, 14
5	8, 9, 20, 22, 23
6	33, 36

For the existence of an r-SOLS(v), we have the necessary condition in Theorem 1.1, i.e., $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$. It is well-known that an SOLS(v) exists if and only if $v \neq 2, 3, 6$ (see, for example, [18]). This solves the case of $r = v^2$. For the case of $r = v$, we take the symmetric Latin square

$$L = (a_{ij}), \quad a_{ij} = i + j \pmod{v}, \quad i, j \in Z_v.$$

It is easily seen that L is a v-SOLS(v), and we have the following theorem.

Theorem 1.2. *There exist v-SOLS(v) for all integer $v > 0$, and v^2-SOLS(v) for all integer $v > 0$, $v \neq 2, 3, 6$.*

So, we can focus on the cases $v + 1 < r < v^2 - 1$ for the existence of an r-SOLS(v).

For small orders, we have the following results.

Theorem 1.3 (*Zhu and Zhang [20]*). *For order $v = 4$, there is only one $r \in [v+1, v^2-1]$, namely $r = 9$, such that an r-SOLS(v) exists.*

For $v = 5$ and $v + 1 < r < v^2 - 1$, there is an r-SOLS(5) for $r \in \{7, 10, 11, 13, 14, 15, 17, 19, 21\}$ only.

For $v = 6$ and $v + 1 < r < v^2 - 1$, there is an r-SOLS(6) for $r \in [8, 31]$ only.

For $v = 7$ and $v + 1 < r < v^2 - 1$, there is an r-SOLS(7) for all $r \in [9, 47] \setminus \{46\}$ only.

r-SOLS(8) for all $r \in [10, 62]$ are listed at the web site http://www.cs.uiowa.edu/~hzhang/sr/. So we have

Theorem 1.4. *There exists an r-SOLS(8) for every $r \in [8, 64] \setminus \{9, 63\}$.*

Zhu and Zhang [20, Conjecture 3.1] conjectured that there is an integer v_0 such that for any $v \geqslant v_0$, there exists an r-SOLS(v) for any $r \in [v, v^2] \setminus \{v+1, v^2-1\}$. The authors [14] have shown that $v_0 \leqslant 28$.

Theorem 1.5 (*Xu and Chang [14, Theorem 6.4]*). *For any integer $v \geqslant 28$, there exists an r-SOLS(v) if and only if $v \leqslant r \leqslant v^2$ and $r \notin \{v+1, v^2-1\}$.*

In this paper, we investigate the existence of r-SOLS(v) for the remaining v, $9 \leqslant v \leqslant 27$.

2. Direct constructions

Let S be a set and L and M be two Latin squares based on S. If the superposition of L and M yields every ordered pair in $S \times S$, then L and M is said to be a pair of mutually orthogonal Latin squares, and denoted by MOLS$(|S|)$, where $|S|$ is the cardinality of S.

Let $\mathscr{H} = \{H_1, H_2, \ldots, H_k\}$ be a set of nonempty subsets of S. A *holey* (or, *incomplete*) *Latin square* having *hole set* $\mathscr{H}$ is an $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties:

(1) every cell of L is either empty or contains a symbol of S,
(2) every symbol of S occurs at most once in any row or column of L,