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# Enumeration of the doubles of the projective plane of order 4[☆]

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**Abstract**

A classification of the doubles of the projective plane of order 4 with respect to the order of the automorphism group is presented and it is established that, up to isomorphism, there are 1 746 461 307 doubles. We start with the designs possessing non-trivial automorphisms. Since the designs with automorphisms of odd prime orders have been constructed previously, we are left with the construction of the designs with automorphisms of order 2. Moreover, we establish that a 2-(21, 5, 2) design cannot be reducible in two inequivalent ways. This makes it possible to calculate the number of designs with only the trivial automorphism, and consequently the number of all double designs. Most of the computer results are obtained by two different approaches and implementations.


*Keywords:* Double design; Projective plane; Construction methods

## 1. Introduction

Let $V=\{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathscr{B}=\{B_j\}_{j=1}^{b}$ a finite collection of $k$-element subsets of $V$, called *blocks*. $D=(V,\mathscr{B})$ is a *design* with parameters $t$-$(v,k,\lambda)$ if any $t$-subset of $V$ is contained in exactly $\lambda$ blocks of $\mathscr{B}$. For the basic concepts and notations concerning combinatorial designs, refer for instance to [1–3,13].

The *incidence matrix* of a design is a (0,1) matrix with $v$ rows and $b$ columns, where the element of the $i$th row and $j$th column is 1 if $P_i \in B_j$ $(i=1,2,\ldots,v;\ j=1,2,\ldots,b)$ and 0 otherwise. The design is completely determined by its incidence matrix.

An *isomorphism* of two designs $D_1=(V_1,\mathscr{B}_1)$ and $D_2=(V_2,\mathscr{B}_2)$ is a bijection between their point sets $V_1$ and $V_2$ and their block collections $\mathscr{B}_1$ and $\mathscr{B}_2$, such that the point-block incidence is preserved. In terms of the incidence

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matrices, two designs are isomorphic if their incidence matrices are equivalent, i.e. if the incidence matrix of the second design can be obtained from the incidence matrix of the first one by a permutation of the rows and columns.

An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that preserves the block collection. The set of all automorphisms of a design forms a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

Each 2-$(v, k, \lambda)$ design determines the existence of 2-$(v, k, m\lambda)$ designs (for any integer $m > 1$), which are called *quasimultiples* of a 2-$(v, k, \lambda)$ design. A quasimultiple 2-$(v, k, m\lambda)$ is *reducible* into $m$ 2-$(v, k, \lambda)$ designs if there is a partition of its blocks into $m$ subcollections, each of which forms a 2-$(v, k, \lambda)$ design. This partition is called a *reduction*. For $m = 2$ quasimultiple designs are called *quasidoubles*, and the reducible quasidouble designs are called *doubles*. We shall denote by $(D_1 \cup D_2)$ a double design which can be reduced to the two designs $D_1$ and $D_2$. A *reduction* of a double design $D$ with parameters $t$-$(v, k, 2\lambda)$ can be represented by a set of two collections of blocks, each containing half of the blocks of $D$, such that each collection of blocks forms a 2-$(v, k, \lambda)$ design. An obvious reduction of a double design $(D_1 \cup D_2)$ is $\{D_1, D_2\}$; the order in which the constituent designs are listed, is not relevant. We will often use the notation $D_2 = \mu D_1$, in which $\mu$ is a point permutation applied to the points of $D_1$ to obtain $D_2$. Doubles can be reducible in more than one way. Two reductions $\{D_1, D_2\}$ and $\{D_3, D_4\}$ of a double design are *equivalent* if and only if there exists some point permutation $\mu$ such that $D_3 = \mu D_1$ and $D_4 = \mu D_2$, or such that $D_4 = \mu D_1$ and$D_3 = \mu D_2$. A double which has, up to equivalence, only one reduction is *uniquely reducible*.

Reducible 2-(21, 5, 2) designs are the subject of the present note, we will show that they are uniquely reducible. Up to equivalence there is a unique 2-(21, 5, 1) design (the projective plane $PG(2, 4)$ of order 4) and the reducible 2-(21, 5, 2) designs are its doubles. The first lower bound on the number of reducible 2-(21, 5, 2) designs is derived in [11] and it is 10. Lower bounds on the number of doubles of projective planes in general are derived in [5,6]. These bounds are much more powerful for projective planes of bigger orders, but for the doubles of the projective plane of order 4 the bound is 24.

In this paper, we enumerate the reducible 2-(21, 5, 2) designs by constructing those which have non-trivial automorphisms, which allows us to calculate the number of all reducible 2-(21, 5, 2). This is possible, because these designs are made up of two 2-(21, 5, 1) subdesigns. For other examples of enumerating designs which contain incidence structures see for instance [8–10,15].

In [14] all 2-(21, 5, 2) designs with automorphisms of odd prime orders were constructed, their number was determined to be 22 998 and 4170 of them were found to be reducible. This leaves only the reducible 2-(21, 5, 2) designs with automorphisms of order 2 to be constructed. There are two types of such automorphisms, namely those which transform each of the constituent 2-(21, 5, 1) designs into itself and those which transform one of the 2-(21, 5, 1) into the other (and vice versa). We construct 40 485 designs of the first type and 991 957 of the second. We study their automorphism groups. The results coincide with those obtained in [14]. Using this data we calculate that the number of all doubles of the projective plane of order 4 is 1 746 461 307.

## 2. Doubles of a uniquely reducible design

Below we will consider doubles of designs for which, up to isomorphism, only one design of its parameter set exists. So instead of $(D_1 \cup D_2)$ we will often use the notation $(D \cup \varphi D)$, where the constituent design $\varphi D$ is obtained from $D$ by a permutation $\varphi$ of its points.

In the rest of this section, $D$ will be a 2-$(v, k, \lambda)$ design and $(D \cup \varphi D)$ will be a uniquely reducible double of $D$. By $G$ we denote the full automorphism group of $D$; by $G_\varphi$ we denote the intersection of the full automorphism groups of $D$ and $\varphi D$; by $\widehat{G}_\varphi$ we denote the full automorphism group of the double design $(D \cup \varphi D)$.

The set of all $v!$ permutations $\varphi$ of the points of $D$ can be partitioned into classes $\mathscr{C}_G(\varphi)$, where $\mathscr{C}_G(\varphi)$ is the set of point permutations $\psi$ having the property that the double $(D \cup \psi D)$ is isomorphic to $(D \cup \varphi D)$. Then obviously

$$v! = \sum_{\mathscr{C}_G(\varphi)} |\mathscr{C}_G(\varphi)|. \tag{1}$$

In the following proposition we determine the size of an isomorphism class $\mathscr{C}_G(\varphi)$ with a given representative point permutation $\varphi$.