Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 306 (2006) 3223–3243
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Composition of Post classes and normal forms of Boolean functions

Miguel Couceiro[a,1], Stephan Foldes[b], Erkko Lehtonen[b,*]

[a]*Department of Mathematics, Statistics and Philosophy, University of Tampere, FI-33014 Tampereen yliopisto, Finland*
[b]*Institute of Mathematics, Tampere University of Technology, P.O. Box 553, FI-33101 Tampere, Finland*

Received 23 March 2005; received in revised form 12 May 2006; accepted 25 June 2006
Available online 6 September 2006

Abstract

The class composition $\mathscr{C} \circ \mathscr{K}$ of Boolean clones, being the set of composite functions $f(g_1, \ldots, g_n)$ with $f \in \mathscr{C}, g_1, \ldots, g_n \in \mathscr{K}$, is investigated. This composition $\mathscr{C} \circ \mathscr{K}$ is either the join $\mathscr{C} \vee \mathscr{K}$ in the Post Lattice or it is not a clone, and all pairs of clones $\mathscr{C}, \mathscr{K}$ are classified accordingly.

Factorizations of the clone Ω of all Boolean functions as a composition of minimal clones are described and seen to correspond to normal form representations of Boolean functions. The median normal form, arising from the factorization of Ω with the clone *SM* of self-dual monotone functions as the leftmost composition factor, is compared in terms of complexity with the well-known DNF, CNF, and Zhegalkin (Reed–Muller) polynomial representations, and it is shown to provide a more efficient normal form representation.

MSC: 06E30; 08A70

Keywords: Function class composition; Clones; Boolean functions; Post classes; Class factorization; Normal forms; DNF; CNF; Zhegalkin polynomial; Reed–Muller polynomial; Formulas; Efficient representations; Complexity; Median; Ternary majority

1. Introduction and notation

Let $\mathbb{B}=\{0,1\}$. A *Boolean function* is a map $f : \mathbb{B}^n \to \mathbb{B}$, for some positive integer n called the *arity* of f. Because we only discuss Boolean functions, we refer to them simply as *functions.* A *class* of functions is a subset $\mathscr{C} \subseteq \bigcup_{n \geqslant 1} \mathbb{B}^{\mathbb{B}^n}$. For a fixed arity n, the n different *projection maps* $(a_1, \ldots, a_n) \mapsto a_i$, $1 \leqslant i \leqslant n$, are also called *variables*, denoted $x_1, \ldots, x_n$, where the arity is clear from the context.

If f is an n-ary function and $g_1, \ldots, g_n$ are all m-ary functions, then the *composition* $f(g_1, \ldots, g_n)$ is an m-ary function, and its value on $(a_1, \ldots, a_m) \in \mathbb{B}^m$ is $f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$. Let $\mathscr{I}$ and $\mathscr{J}$ be classes of functions. The *composition of* $\mathscr{I}$ *with* $\mathscr{J}$, denoted $\mathscr{I} \circ \mathscr{J}$ (or sometimes, when the context is clear, just $\mathscr{I}\mathscr{J}$),

* Corresponding author.
E-mail addresses: miguel.couceiro@uta.fi (M. Couceiro), stephan.foldes@tut.fi (S. Foldes), erkko.lehtonen@tut.fi (E. Lehtonen).
[1] Research partially supported by the Graduate School in Mathematical Logic MALJA and by grant #28139 from the Academy of Finland.

0012-365X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2006.06.014

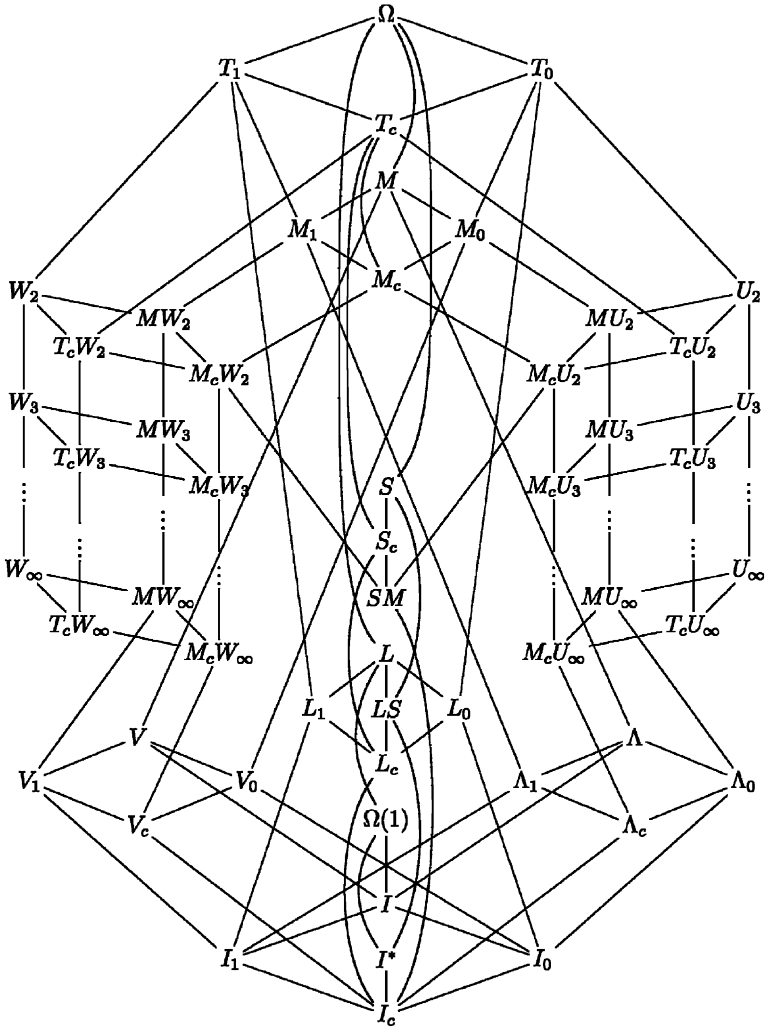

Fig. 1. Post Lattice.

is defined as

$$\mathscr{I} \circ \mathscr{J} = \{f(g_1, \ldots, g_n) : n, m \geqslant 1,\ \ f \ n\text{-ary in } \mathscr{I},\ \ g_1, \ldots, g_n \ m\text{-ary in } \mathscr{J}\}.$$

A *clone* is a class $\mathscr{C}$ of functions that contains all projections and satisfies $\mathscr{C}\mathscr{C} \subseteq \mathscr{C}$ (or equivalently, $\mathscr{C}\mathscr{C} = \mathscr{C}$).

The clones of Boolean functions, originally described by Post [14] (see [16,18,22]) for shorter recent proofs), form an algebraic lattice, where the lattice operations are the following: meet is the intersection, join is the smallest clone that contains the union. The greatest element is the clone Ω of all Boolean functions; the least element is the clone I_c of all projections. These clones and the lattice are often called the Post classes and the Post Lattice, respectively. For the nomenclature of the Post classes, see Appendix. The Post Lattice is illustrated in Fig. 1.

The set $\mathbb{B}^n$ is a Boolean (distributive and complemented) lattice of 2^n elements under the component-wise order of vectors. We will write simply $\mathbf{a} \preccurlyeq \mathbf{b}$ to denote comparison in this lattice. The *complement* of a vector $\mathbf{a} = (a_1, \ldots, a_n)$ is defined as $\overline{\mathbf{a}} = (1 - a_1, \ldots, 1 - a_n)$. We denote $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$. Vectors are also called *points*.

The set $\mathbb{B}^{\mathbb{B}^n}$ is a Boolean lattice of 2^{2^n} elements under the point-wise ordering of functions. Both $\mathbb{B}^n$ and $\mathbb{B}^{\mathbb{B}^n}$ are vector spaces over the two-element field $\mathrm{GF}(2) = \mathbb{B}$.

For a function f, the *dual* of f is defined as $f^{\mathrm{d}}(\mathbf{a}) = \overline{f(\overline{\mathbf{a}})}$ for all $\mathbf{a}$. For a class $\mathscr{C}$, the *dual* of $\mathscr{C}$ is defined as $\mathscr{C}^{\mathrm{d}} = \{f^{\mathrm{d}} : f \in \mathscr{C}\}$. The dual of a clone is a clone, and it is well-known that dualization gives the only nontrivial order-automorphism of the Post Lattice.