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Note

On hyperfocused arcs in $PG(2, q)$ ☆

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Abstract

A k-arc in a Dearguesian projective plane whose secants meet some external line in $k-1$ points is said to be hyperfocused. Hyperfocused arcs are investigated in connection with a secret sharing scheme based on geometry due to Simmons. In this paper it is shown that point orbits under suitable groups of elations are hyperfocused arcs with the significant property of being contained neither in a hyperoval nor in a proper subplane. Also, the concept of generalized hyperfocused arc, i.e. an arc whose secants admit a blocking set of minimum size, is introduced: a construction method is provided, together with the classification for size up to 10.

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1. Introduction

Hyperfocused arcs were introduced in connection with a secret sharing scheme based on geometry due to Simmons [11]. The implementation of this scheme needs an arc in a Desarguesian projective plane with the property that its secant lines intersect some external line in a minimal number of points. Simmons only considered planes of odd order, where this minimal number equals the number of points of the arc [4]. He introduced the term sharply focused set for arcs satisfying the aforementioned property. Sharply focused sets in Desarguesian projective planes of odd order were classified by Beutelspacher and Wettl [3], whose result was based on a previous paper by Wettl [12].

In 1997 Holder [9] extended Simmons's investigation to Desarguesian planes of even order. In such planes the secants of an arc of size k may meet an external line in only $k-1$ points, yet the classification of arcs having this property seems to be an involved problem. Holder used the term supersharply focused sets for such arcs and gave some constructions for them.

In a recent paper [5], Cherowitzo and Holder proposed the term hyperfocused arc instead of supersharply focused set. They provided the classification of small hyperfocused arcs, and constructed new examples, one of which gave a negative answer to a question raised by Drake and Keating [7] on the possible sizes of a hyperfocused arc.

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Some open problems were pointed out by Cherowitzo and Holder, including the existence of hyperfocused arcs which are neither contained in a proper subplane nor in a hyperoval. In this paper a positive answer to this question is given. The main tool is the investigation of the so-called translation arcs, i.e. arcs which are point orbits under a group of elations. In Section 3 it is shown that such arcs are hyperfocused, and it is proved that sometimes they are contained neither in a hyperoval nor in a proper subplane, see Theorem 3.7.

The concept of hyperfocused arc can be naturally extended to that of generalized hyperfocused arc, that is an arc of size k for which there exists an external point set of size $k-1$ meeting each of its secants. Recently, Aguglia et al. [1] proved that in Desarguesian planes of even order any generalized hyperfocused arc is hyperfocused, provided that it is contained in a conic. In Section 4 we provide a construction of generalized hyperfocused arcs which are not hyperfocused. Also, a classification of small generalized hyperfocused arcs is proved using the graph-theoretic concept of 1-factorizations of a complete graph, see Section 5.

2. Definitions and notation

Let $PG(2,q)$ be the Desarguesian plane over $\mathbb{F}_q$, the finite field with q elements. A k-arc $\mathscr{K}$ in $PG(2,q)$ is a set of k points no three of which are collinear. Any line containing two points of $\mathscr{K}$ is said to be a *secant* of $\mathscr{K}$. A *blocking set of the secants of* $\mathscr{K}$ is a point set $\mathscr{B} \subset PG(2,q)\backslash\mathscr{K}$ having non-empty intersection with each secant of $\mathscr{K}$. As the number of secants of $\mathscr{K}$ is $k(k-1)/2$, the size of $\mathscr{B}$ is at least $k-1$. If this lower bound is attained, $\mathscr{B}$ is said to be *of minimum size*. Also, $\mathscr{B}$ is *linear* if it is contained in a line.

Arcs in $PG(2,q)$ admitting a linear blocking set of minimum size of their secants are called *hyperfocused arcs*. As mentioned in the Introduction, hyperfocused arcs exist only in $PG(2,q)$ for q even. Therefore in the whole paper we assume $q=2^r$.

Throughout, we fix the following notation. Let (X_1,X_2,X_3) be homogeneous coordinates for points in $PG(2,q)$, and let ℓ_∞ be the line of equation $X_3=0$. Given a pair $A=(a,b)$ in $\mathbb{F}_q\times\mathbb{F}_q$, denote $\overline{A}$ the point in $PG(2,q)$ with coordinates $(a,b,1)$, and $\overline{A}_\infty$ the point $(a,b,0)$. Also, let φ_A be the projectivity

$$\varphi_A : (X_1,X_2,X_3) \mapsto (X_1+a_1X_3, X_2+a_2X_3, X_3).$$

Clearly, φ_A is an elation with axis ℓ_∞, and conversely for any non-trivial elation φ with axis ℓ_∞ there exists $A\in\mathbb{F}_q\times\mathbb{F}_q$, $A\neq(0,0)$, such that $\varphi=\varphi_A$.

Given an additive subgroup G of $\mathbb{F}_q\times\mathbb{F}_q$, let $\mathscr{K}_G(P)$ be the orbit of the point $P\in PG(2,q)\backslash\ell_\infty$ under the action of the group

$$T_G := \{\varphi_A \mid A\in G\}.$$

Clearly, any two orbits $\mathscr{K}_G(P)$ and $\mathscr{K}_G(Q)$ with $P,Q\in PG(2,q)\backslash\ell_\infty$ are projectively equivalent. For brevity, write $\mathscr{K}_G$ for $\mathscr{K}_G(O)$, where $O=(0,0,1)$. Note that

$$\mathscr{K}_G := \{\overline{A} \mid A\in G\}.$$

A k-arc in $PG(2,q)$ coinciding with $\mathscr{K}_G(P)$ for some additive subgroup $G\subset\mathbb{F}_q\times\mathbb{F}_q$ and some $P\in PG(2,q)\backslash\ell_\infty$ will be called a *translation arc*.

3. Translation arcs

The following proposition shows that any translation arc is a hyperfocused arc.

Proposition 3.1. *Let $\mathscr{K}$ be a translation arc. Then there exists a blocking set of the secants of $\mathscr{K}$ of minimum size which is contained in ℓ_∞.*

Proof. Let G be an additive subgroup of $\mathbb{F}_q\times\mathbb{F}_q$ such that $\mathscr{K}$ is projectively equivalent to $\mathscr{K}_G$. To prove the assertion, it is enough to show that every secant of $\mathscr{K}$ meets ℓ_∞ in a point $\overline{C}_\infty$ for some $C\in G\backslash\{(0,0)\}$. For $A,B\in G$, $A\neq B$, let l_{AB} be the secant of $\mathscr{K}$ passing through $\overline{A}$ and $\overline{B}$. The intersection point of l_{AB} and ℓ_∞ is $\overline{(A+B)}_\infty$. Then the claim is proved, as $A+B$ is a non-zero element of G. $\square$