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Note

Hamilton cycles in digraphs of unitary matrices

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Abstract

A set $S \subseteq V$ is called a q^+-*set* (q^--*set*, respectively) if S has at least two vertices and, for every $u \in S$, there exists $v \in S$, $v \neq u$ such that $N^+(u) \cap N^+(v) \neq \emptyset$ ($N^-(u) \cap N^-(v) \neq \emptyset$, respectively). A digraph D is called *s-quadrangular* if, for every q^+-set S, we have $|\cup\{N^+(u) \cap N^+(v) : u \neq v, u, v \in S\}| \geqslant |S|$ and, for every q^--set S, we have $|\cup\{N^-(u) \cap N^-(v) : u, v \in S)\}| \geqslant |S|$. We conjecture that every strong s-quadrangular digraph has a Hamilton cycle and provide some support for this conjecture.

Keywords: Digraph; Hamilton cycle; Sufficient conditions; Conjecture; Quantum mechanics; Quantum computing

1. Introduction

The hamiltonian cycle problem is one of the central problems in graph theory and its applications [2,6,13]. Many sufficient conditions were obtained for hamiltonicity of undirected graphs [6] and only a few such conditions are proved for directed graphs (for results and conjectures on sufficient conditions for hamiltonicity of digraphs, see [2]). This indicates that the asymmetry of the directed case makes the hamilton cycle problem significantly harder, in a sense.

For a digraph $D = (V, A)$ and $x \neq y \in V$, we say that x *dominates* y, denoted $x \to y$, if $xy \in A$. All vertices dominated by x are called the *out-neighbors* of x; we denote the set of out-neighbors by $N^+(x)$. All vertices that dominate x are *in-neighbors* of x; the set of in-neighbors is denoted by $N^-(x)$. A set $S \subseteq V$ is called a q^+-*set* (q^--*set*, respectively) if S has at least two vertices and, for every $u \in S$, there exists $v \in S, v \neq u$, such that $N^+(u) \cap N^+(v) \neq \emptyset$ ($N^-(u) \cap N^-(v) \neq \emptyset$, respectively). A digraph D is called *s-quadrangular* if, for every q^+-set S, we have $|\cup\{N^+(u) \cap N^+(v) : u \neq v, u, v \in S\}| \geqslant |S|$ and, for every q^--set S, we have $|\cup\{N^-(u) \cap N^-(v) : u, v \in S)\}| \geqslant |S|$. A digraph D is *strong* if there is a path from x to y for every ordered pair x, y of vertices of D.

We believe that the following claim holds:

Conjecture 1.1. Every strong s-quadrangular digraph is hamiltonian.

A complex $n \times n$ matrix U is *unitary* if $U \cdot U^\dagger = U^\dagger \cdot U = I_n$, where $U^\dagger$ denotes the conjugate transpose of U and I_n the $n \times n$ identity matrix. The *digraph of* an $n \times n$ matrix M (over any field) is a digraph on n vertices with an

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arc ij if and only if the (i, j)-entry of the M is non-zero. It was shown in [12] that the digraph of a unitary matrix is s-quadrangular; s-quadrangular tournaments were studied in [10].

It follows that if Conjecture 1.1 is true, then the digraph of an irreducible unitary matrix is hamiltonian. Unitary matrices are important in quantum mechanics and, at present, are central in the theory of quantum computation [11]. In particular, we may associate a strong digraph to a quantum system whose unitary evolution allows transitions only along the arcs of the digraph (that is, respecting the topology of the graph, like in discrete quantum walks [1,9]). Then, if the conjecture is true, the digraph would be necessarily hamiltonian. Moreover, the conjecture is important in the attempt to understand the combinatorics of unitary and unistochastic matrices, see, e.g., [3,4,14]. If the conjecture is true, then the digraph of an irreducible weighing matrix has a hamilton cycle (see [5] for a reference on weighing matrices). Also, since the kronecker product of unitary matrices preserves unitarity, if K and H are digraphs of irreducible unitary matrices, then their kronecker product $K \otimes H$ (see [8] for an interesting collection of notions and results on graph products) has a hamilton cycle provided $K \otimes H$ is strong. The *complete biorientation* of an undirected graph G is a digraph obtained from G by replacing every edge xy by the pair xy, yx of arcs. A graph is *s-quadrangular* if its complete biorientation is s-quadrangular. Certainly, the following is a weakening of Conjecture 1.1:

Conjecture 1.2. Every connected s-quadrangular graph is hamiltonian.

In this paper, we provide some support to the conjectures. In Section 2, we show that if a strong s-quadrangular digraph D has the maximum semi-degree at most 3, then D is hamiltonian. In our experience, to improve the result by replacing $\Delta^0(D)\leqslant 3$ with $\Delta^0(D)\leqslant 4$ appears to be a very difficult task. In Section 3, we show the improved result only for the case of undirected graphs. Even in this special case the proof is fairly non-trivial. Before recalling some standard definitions and proving our results, it is worth mentioning that the line digraphs of eulerian digraphs are all s-quadrangular and hamiltonian. We have verified Conjecture 1.1 for all digraphs with at most five vertices and a number of digraphs with six vertices.

The number of out-neighbors (in-neighbors) of x is the *out-degree* $d^+(x)$ of x (*in-degree* $d^-(x)$ of x). The maximum *semi-degree* $\Delta^0(D) = \max\{d^+(x), d^-(x) : x \in V\}$. A collection of disjoint cycles that include all vertices of D is called a *cycle factor* of D. We denote a cycle factor as the union of cycles $C_1 \cup \cdots \cup C_t$, where the cycles C_i are disjoint and every vertex of D belongs to a cycle C_j. If $t = 1$, then clearly C_1 is a *hamilton cycle* of D. A digraph with a hamilton cycle is called *hamiltonian*. Clearly, the existence of a cycle factor is a necessary condition for a digraph to be hamiltonian.

2. Supporting Conjecture 1.1

The existence of a cycle factor is a natural necessary condition for a digraph to be hamiltonian [7]. The following necessary and sufficient conditions for the existence of a cycle factor is well known, see, e.g., [2, Proposition 3.11.6].

Lemma 2.1. *A digraph H has a cycle factor if and only if, for every $X \subseteq V(H)$, $|\bigcup_{x\in X} N^+(x)|\geqslant|X|$ and $|\bigcup_{x\in X} N^-(x)|\geqslant|X|$.*

Using this lemma, it is not difficult to prove the following theorem:

Theorem 2.2. *Every strong s-quadrangular digraph $D = (V, A)$ has a cycle factor.*

Proof. Let $X \subseteq V$. If X is a q^+-set, then

$$|X|\leqslant \left|\bigcup(N^+(u)\cap N^+(v) : u \neq v, u, v \in X)\right| \leqslant \left|\bigcup_{x\in X} N^+(x)\right|.$$

If X is not a q^+-set, then consider a maximal subset S of X, which is a q^+-set (possibly $S=\emptyset$). Since D is strong every vertex of X dominates a vertex. Moreover, since every vertex of $X - S$ dominates a vertex that is not dominated by