Available online at www.sciencedirect.com
SCIENCE DIRECT
Discrete Mathematics 306 (2006) 351 – 358
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A linear time algorithm to list the minimal separators of chordal graphs ☆

L. Sunil Chandran[1], Fabrizio Grandoni

Max-Planck Institut für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany

Received 10 November 2003; received in revised form 16 December 2004; accepted 8 November 2005
Available online 27 January 2006

Abstract

Kumar and Madhavan [Minimal vertex separators of chordal graphs, Discrete Appl. Math. 89 (1998) 155–168] gave a linear time algorithm to list all the minimal separators of a chordal graph. In this paper we give another linear time algorithm for the same purpose. While the algorithm of Kumar and Madhavan requires that a specific type of PEO, namely the MCS PEO is computed first, our algorithm works with any PEO. This is interesting when we consider the fact that there are other popular methods such as Lex BFS to compute a PEO for a given chordal graph.

Keywords: Minimal separator; Chordal graph; Perfect elimination ordering

1. Introduction

Let C be a cycle in a graph G. A chord of C is an edge of G joining two vertices of C which are not consecutive. A graph G is called a chordal (or triangulated) graph iff every cycle in G, of length 4 or more has a chord. Chordal graphs arise in many applications (see [7,13,17]). Chordal graphs constitute one of the most important subclasses of perfect graphs [7].

In a connected graph G, a *separator* S is a subset of vertices whose removal separates G into at least two connected components. S is called a $(a-b)$ *separator* iff it disconnects vertices a and b. A $(a-b)$ separator is said to be a *minimal separator* iff it does not contain any other $(a-b)$ separator.

The problem of listing all minimal separators is one of the fundamental enumeration problems in graph theory, which has great practical importance in reliability analysis for networks and operations research for scheduling problems [8,6,1].

The problem of listing all minimal separators of an undirected graph is considered by various authors [6,11,15]. A $O(n^6 R_\Sigma)$ algorithm is given in [11], to list all *minimal* separators, where R_Σ is the total number of minimal separators in the graph. This is improved in [15] to $O(n^3 R_\Sigma^+ + n^4 R_\Sigma)$, where $R_\Sigma^+ \leqslant (n(n-1)/2 - m) R_\Sigma$. ($n$ and m represent the

☆ A preliminary version of this paper appeared in the Proceedings of the Seventh International Computing and Combinatorics Conference 2001, Lecture Notes in Computer Science, vol. 2108, Springer, Berlin.

E-mail addresses: sunil@mpi-sb.mpg.de (L.S. Chandran), grandoni@mpi-sb.mpg.de (F. Grandoni).

[1] Part of this work was done when the author was at the Indian Institute of Science, Bangalore and was partially supported by the INFOSYS fellowship.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.12.010

number of vertices and number of edges respectively.) The current best-time algorithm for this problem is by Berry et al. [2]: they present an algorithm which computes the set of minimal separators of a graph in $O(n^3 R_\Sigma)$ time. Algorithms to list the minimal separators for some subclasses of perfect graphs (e.g. permutation graphs) are given in [9,10].

Kumar and Madhavan [12] presented a linear time ($O(m+n)$) algorithm that lists all minimal separators of a chordal graph. Their algorithm first computes a specific kind of perfect elimination ordering (namely the ordering given by the maximum cardinality search (MCS) algorithm of Yannakakis and Tarjan [18]) and then makes use of certain properties of this particular PEO to list all the minimal separators. But there exists many other ways to generate PEOs. For example, the Lexico Graphic Breadth First Search algorithm of Rose et al. [14] can output a PEO which is different from what is generated by MCS. In fact even the Lex BFS and MCS together also cannot exhaust all the possible PEOs. Shier [16] gives a characterization of all the possible PEOs in a chordal graph. Chandran et al. [4] give a fast algorithm for generating all the PEO in a given chordal graphs in constant amortized time.

In this paper we give a different linear time algorithm for listing all the minimal separators of a chordal graph. The advantage of this algorithm over the algorithm of Kumar and Madhavan is that, it does not depend on the particular type of PEO used. For example, there may be an application using the Lex BFS PEO and then at some point if it wants to list the minimal separators, it is a waste of effort to recompute a MCS PEO just for this purpose.

Our algorithm is based on the same structural characterization of minimal separators of chordal graphs as that of [12]. But the algorithms are different. (In fact when we wrote the preliminary version of the paper, we were not aware of Kumar and Madhavan's work, and thus a different proof of this structural characterization (Theorem 1) also appears in the preliminary version.)

2. Preliminaries

Let $G=(V,E)$ be a simple, connected, undirected graph. $|V|$ and $|E|$ will be denoted by n and m, respectively. $N(v)$ will denote the set of neighbours of v, that is $N(v)=\{u \in V : (u,v) \in E\}$. For $A \subseteq V$, we use $N(A)$ to denote the set $\bigcup_{v \in A} N(v) - A$. The subgraph of G induced by the nodes in A will be denoted by $G[A]$.

A bijection $f : V \to \{1,2,\dots,n\}$ is called an ordering of the vertices of G. Then $f(v)$ is referred to as the number associated with the vertex v, or simply the *number of* v with respect to the ordering f. Given an ordering f of a graph G, we define the following terms.

Definition 1. Let $A \subseteq V$. The *highest*(A) is defined to be the vertex with the highest number in A. Similarly *lowest*(A) is the vertex in A with the lowest number.

Definition 2. A path $P=(w_1,w_2,\dots,w_k)$ in G is called an *increasing path*, iff $f(w_1)<f(w_2)<\cdots<f(w_k)$. It is called a *decreasing path* iff $f(w_1)>f(w_2)>\cdots>f(w_k)$. A single node can be considered as either increasing or decreasing.

Definition 3. A vertex $u \in N(v)$ is called a *higher neighbour* of v iff $f(u)>f(v)$. The set of higher neighbours of v will be denoted by $N_h(v)$ i.e.,

$$N_h(v)=\{u \in N(v) : f(u)>f(v)\}.$$

Similarly, the set of *lower neighbours* of v is denoted by $N_l(v)$.

$$N_l(v)=\{u \in N(v) : f(u)<f(v)\}$$

$d_h(v)=|N_h(v)|$ and $d_l(v)=|N_l(v)|$.

Definition 4. An ordering f of G is called a *perfect elimination ordering* (PEO) iff for each $v \in V$, $G(\{v\} \cup N_h(v))$ is a complete subgraph (clique) of G.

A graph G is chordal if and only if there exists a PEO for G [7]. Note that there can be more than one PEO for a given chordal graph. The observations and the algorithm presented in this paper are valid with respect to any PEO.